CS103
Winter 2026

Lecture 09:

Graph Theory (Part 1 of 3)

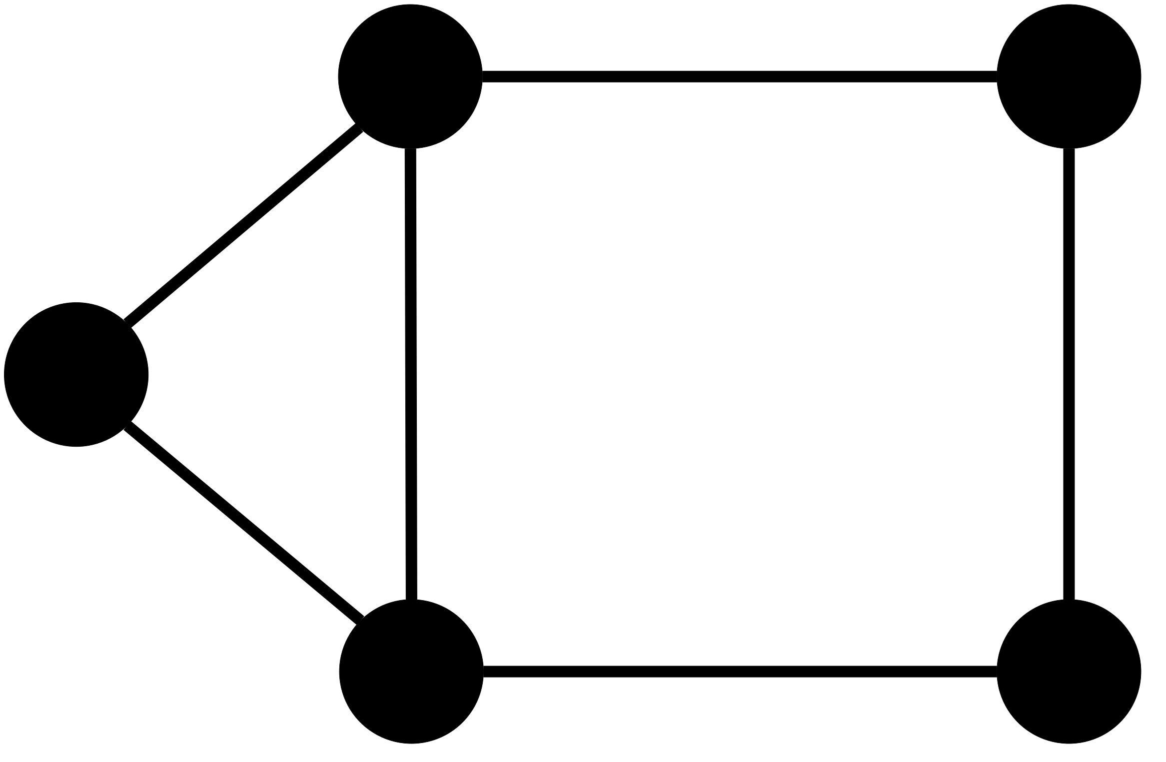
Graphs

Part 1

1. **Graph Overview**
2. Examples
3. Graphs and Digraphs
4. Formalizing Graphs
5. Announcements
6. Vertex Covers
7. Independent Sets
8. A Proof on Graphs
9. Recap and What's Next?

Graph Overview

A ***graph*** is a mathematical structure
for representing relationships.



A graph consists of a set of ***nodes*** (or ***vertices***)
connected by ***edges*** (or ***arcs***)

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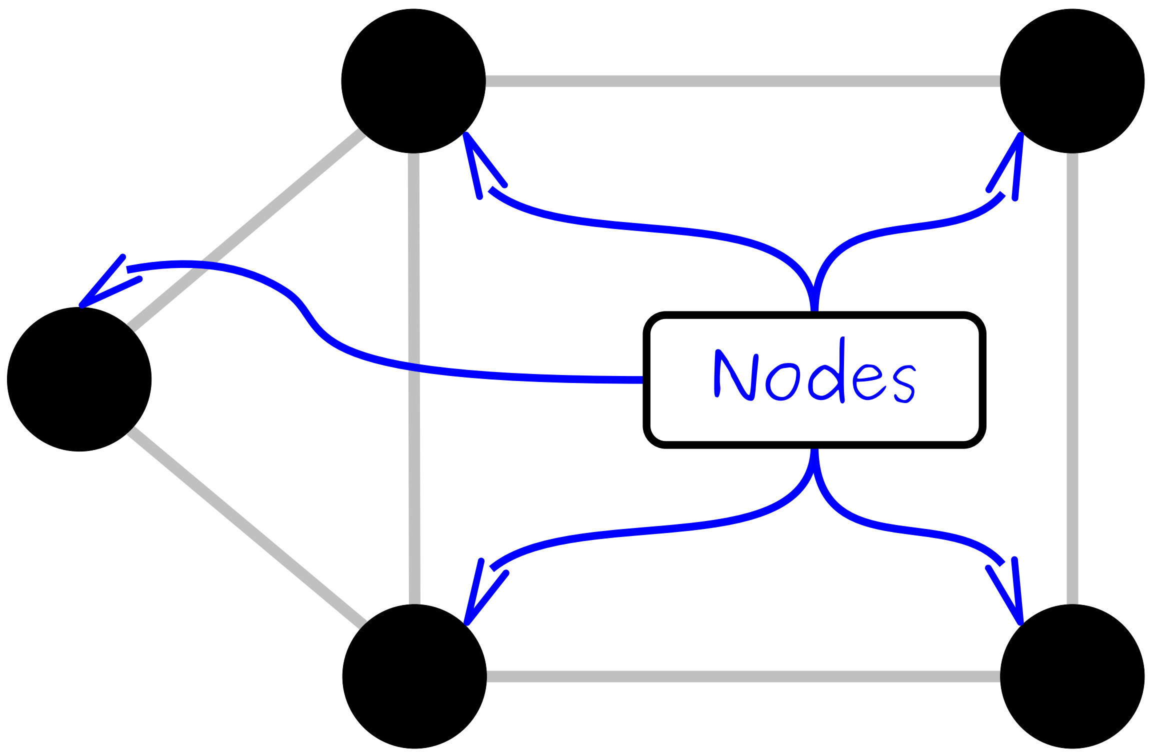

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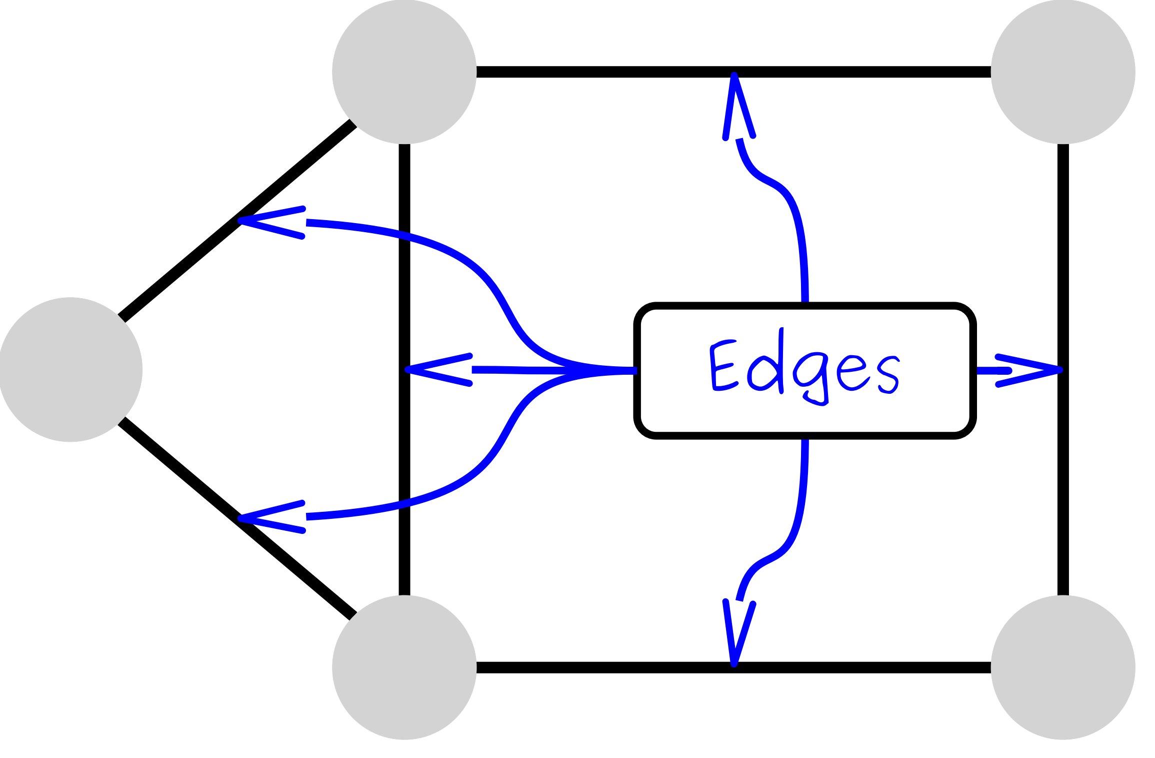

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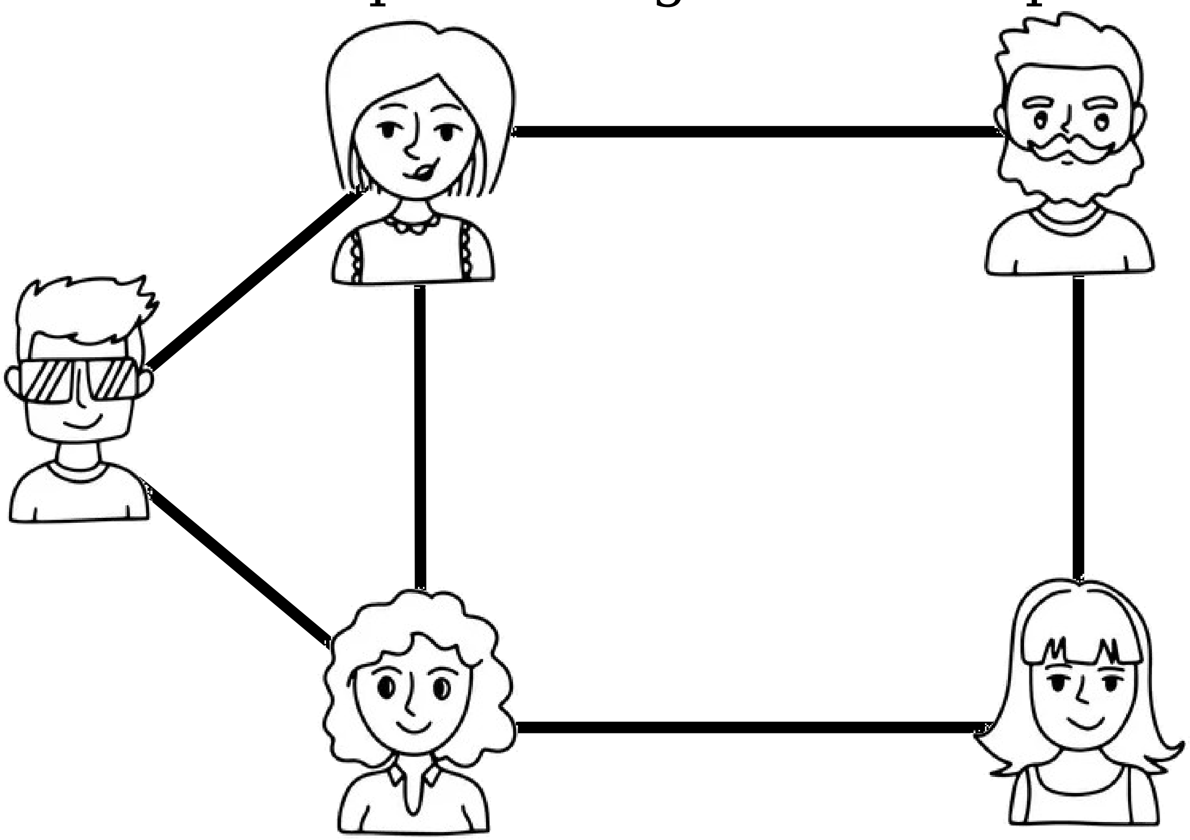

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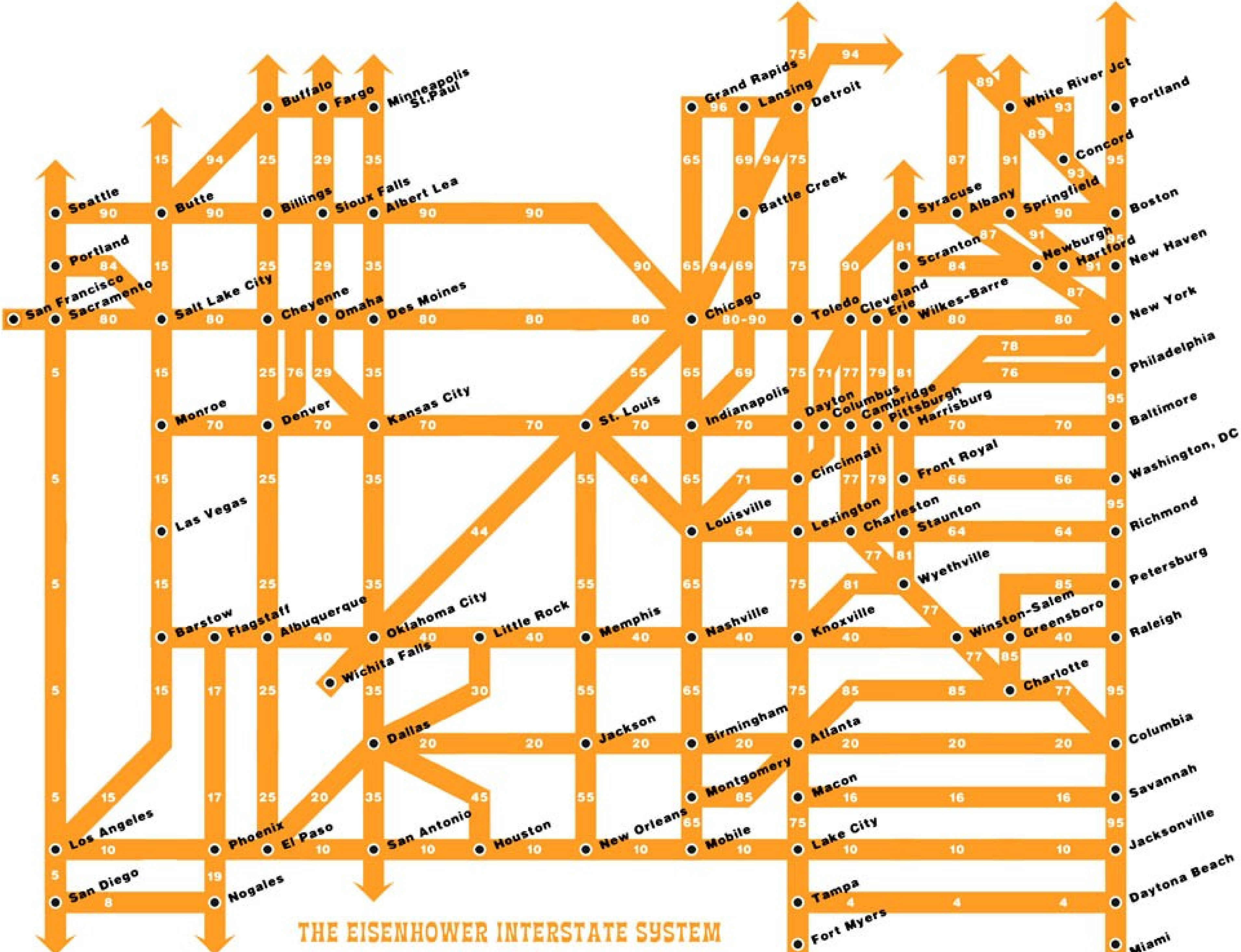


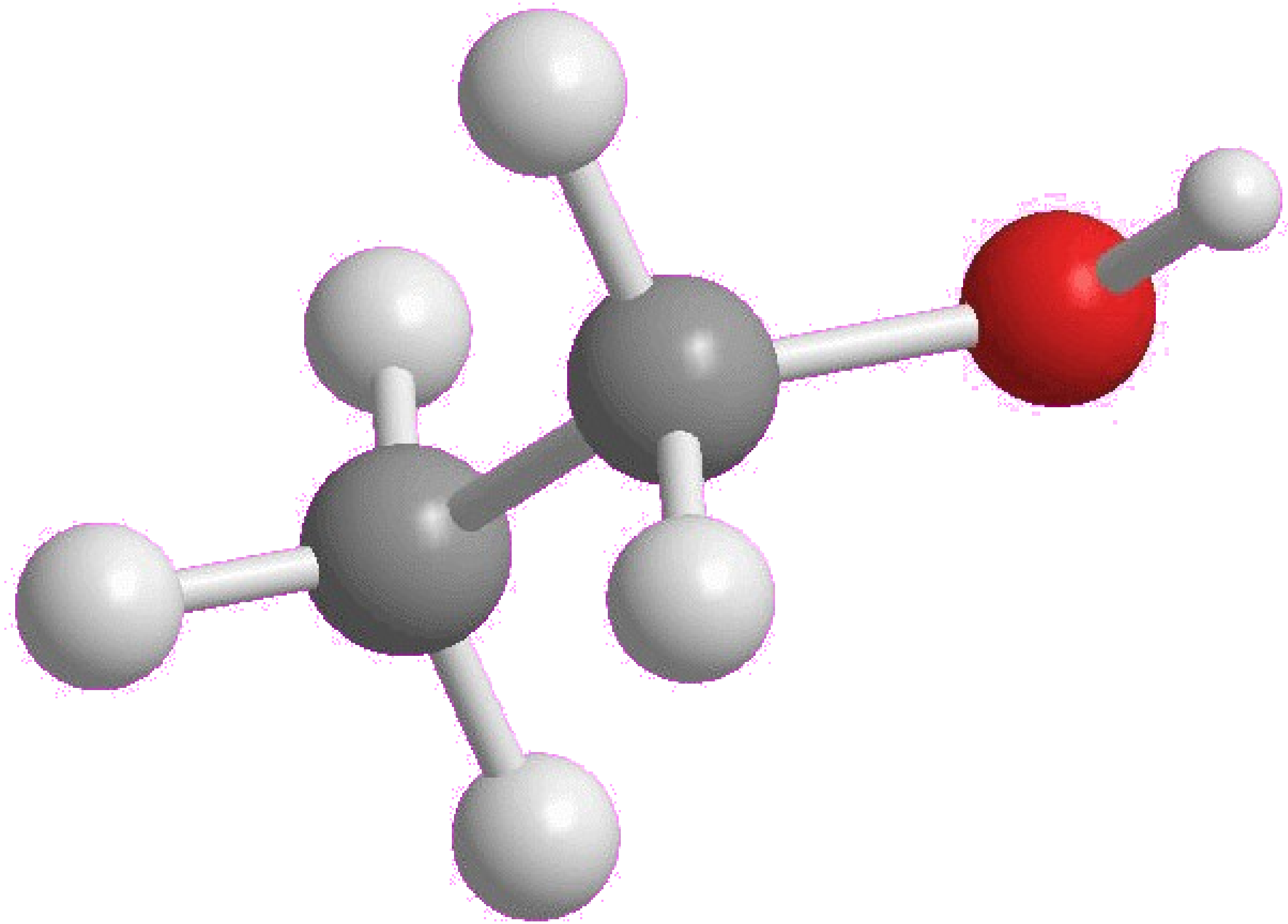
THE EISENHOWER INTERSTATE SYSTEM

(simplified)

CHRIS YATES 2007

http://strangemaps.files.wordpress.com/2007/02/fullinterstatemap-web.jpg

Chemical Bonds



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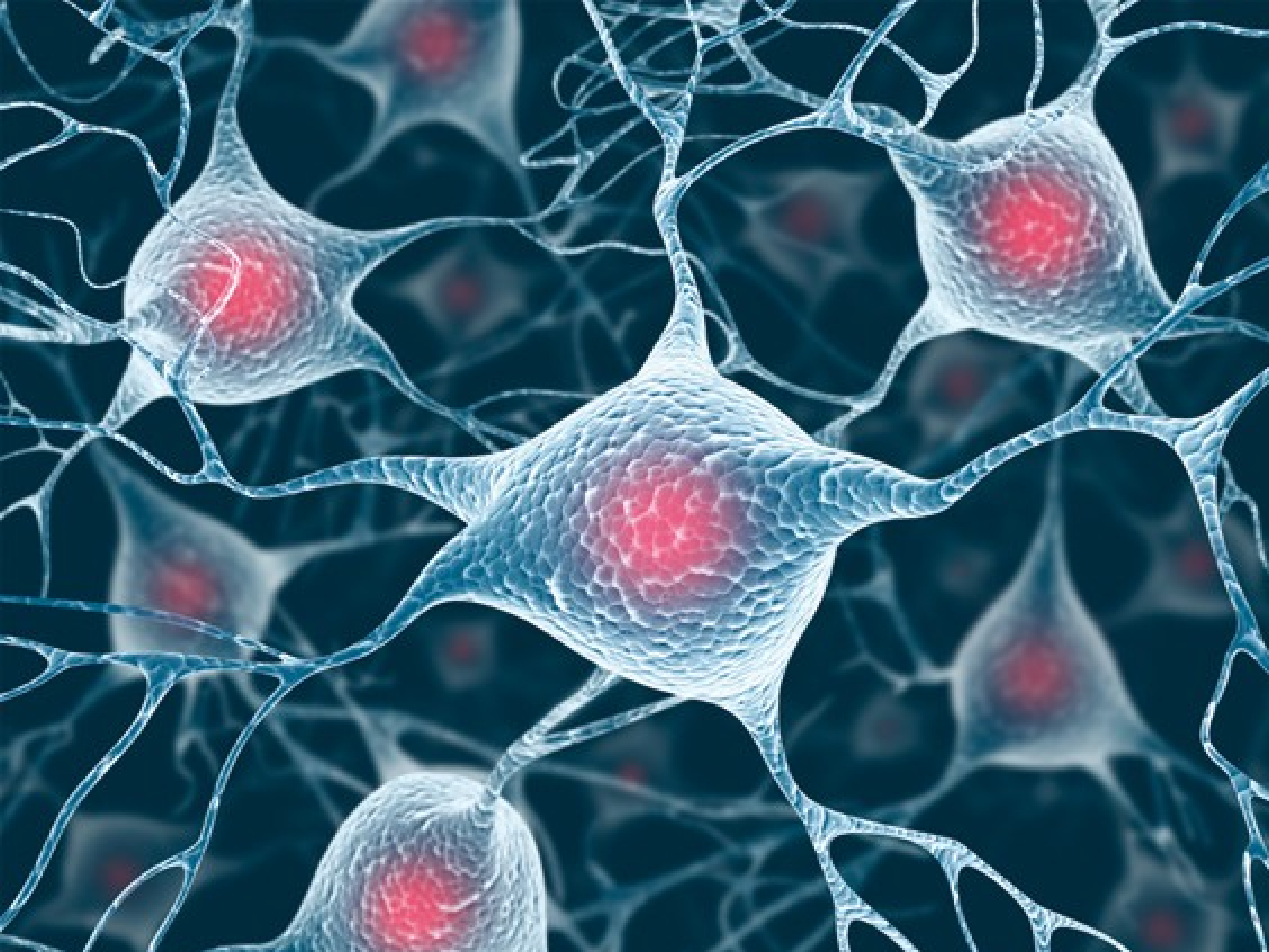

Commonalities

- Each of these structures consists of
  - a collection of objects and
  - links between those objects.
- ***Today's Goal:*** Develop a general framework for describing structures like these that generalizes the idea across a wide domain.

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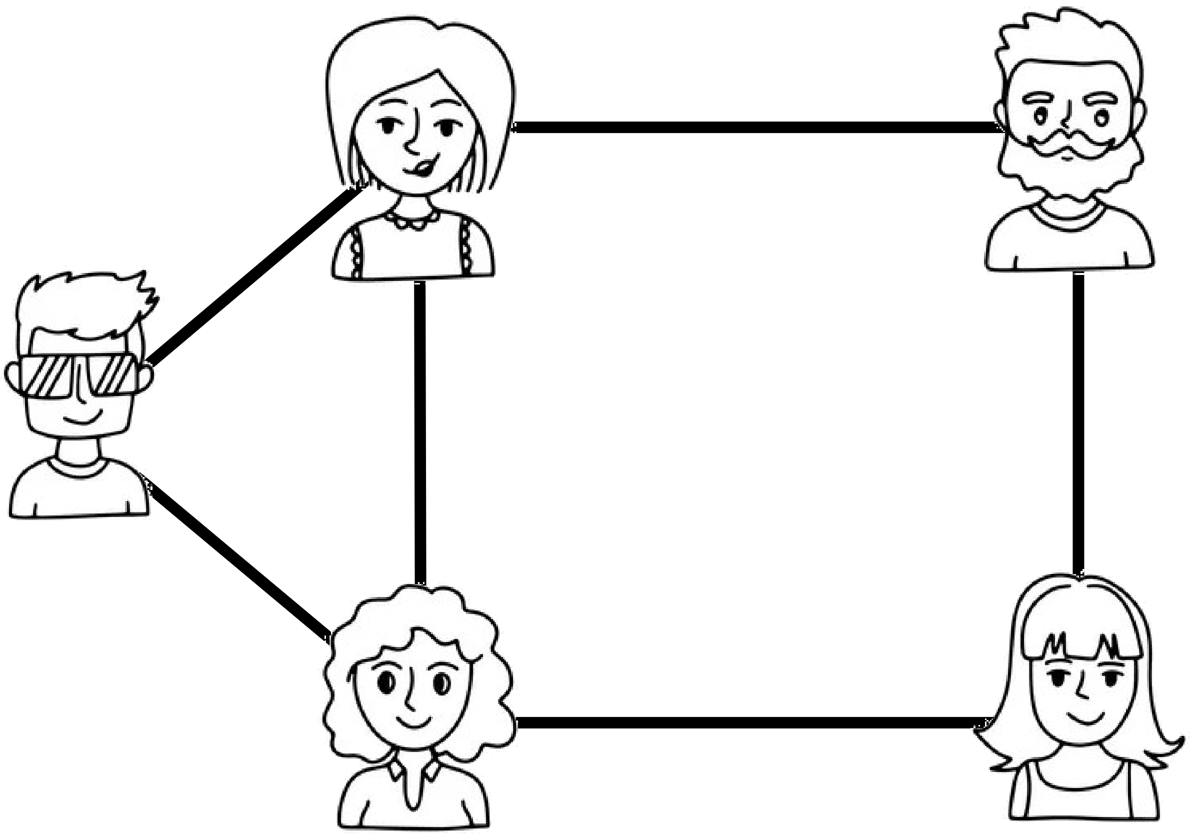
Graphs and Digraphs

Some graphs are ***undirected***.



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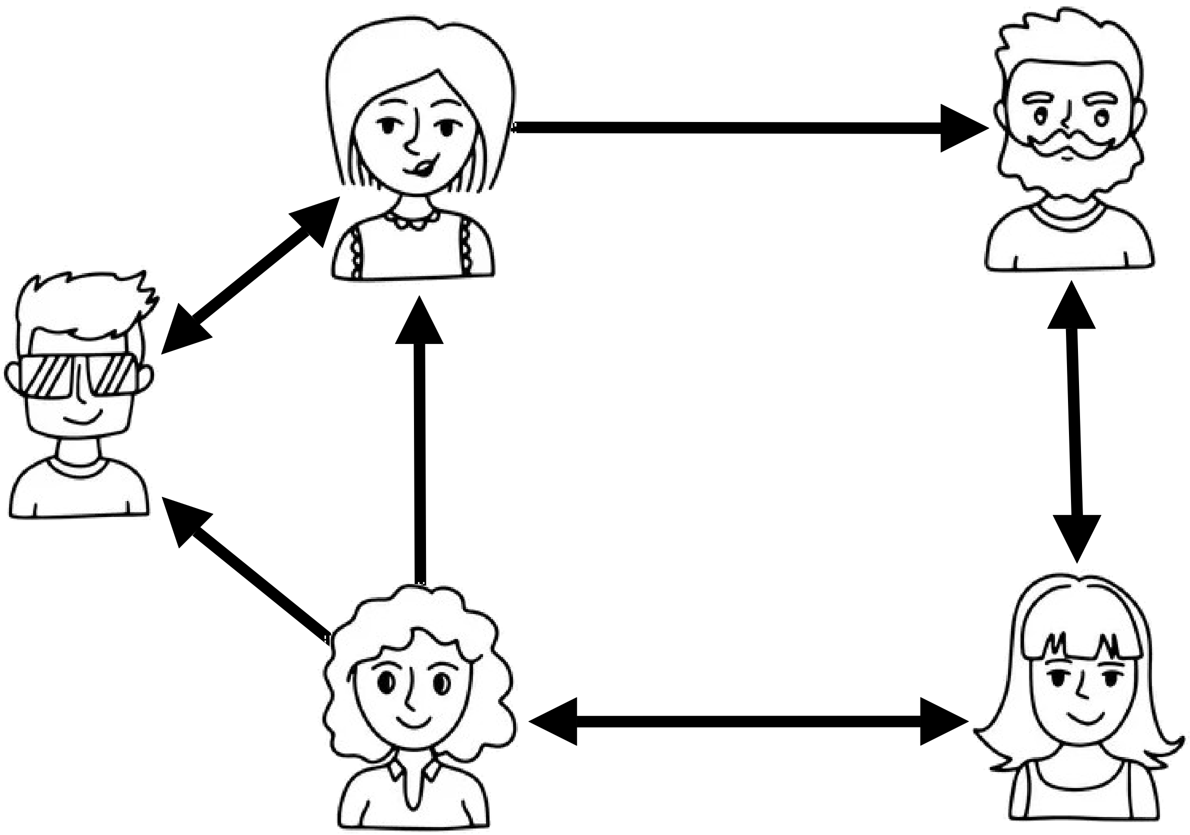
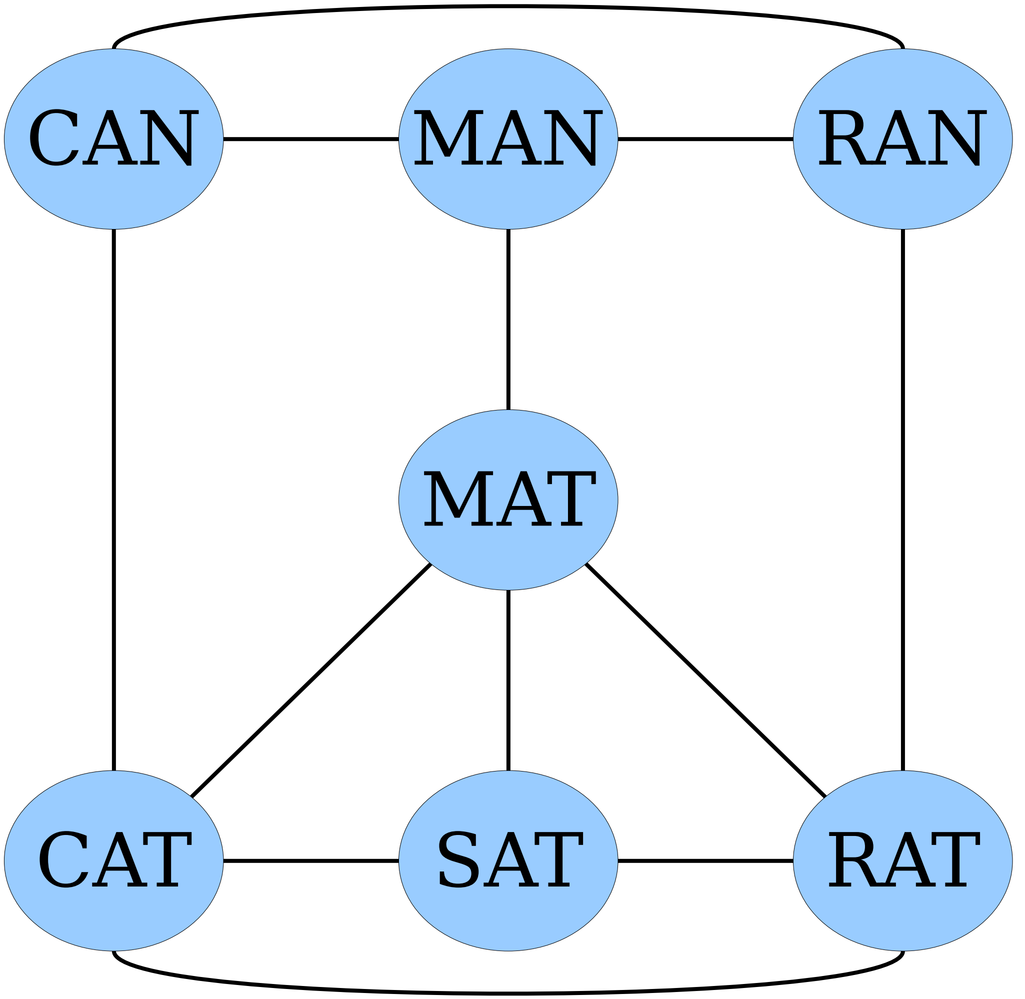


Some graphs are ***directed***.

Graphs and Digraphs

Should this graph be ***directed*** or ***undirected***?



Graphs and Digraphs

- An ***undirected graph*** is one where edges link nodes, with no endpoint preferred over the other.
- A ***directed graph*** (or ***digraph***) is one where edges have an associated direction.
  - (There's something called a ***mixed graph*** that allows for both types of edges, but they're fairly uncommon and we won't talk about them.)
- Unless specified otherwise:

☞ ***"Graph" means "undirected graph"*** ☜

Graphs

Part 1

Formalizing Graphs

- How might we define a graph mathematically?
- We need to specify
  - what the nodes in the graph are, and
  - which edges are in the graph.
- The nodes can be pretty much anything.
- This is pretty broad, but that's a good thing!
- What about edges?

Formalizing Graphs

- An ***unordered pair*** is a set $\{a, b\}$ of two elements $a \neq b$. (Remember that sets are unordered.)
  - For example, $\{0, 1\} = \{1, 0\}$
- An ***undirected graph*** is an ordered pair $G = (V, E)$, where
  - V is a set of nodes, which can be anything, and
  - E is a set of edges, which are *unordered* pairs of nodes drawn from V.
- A ***directed graph*** (or ***digraph***) is an ordered pair $G = (V, E)$, where
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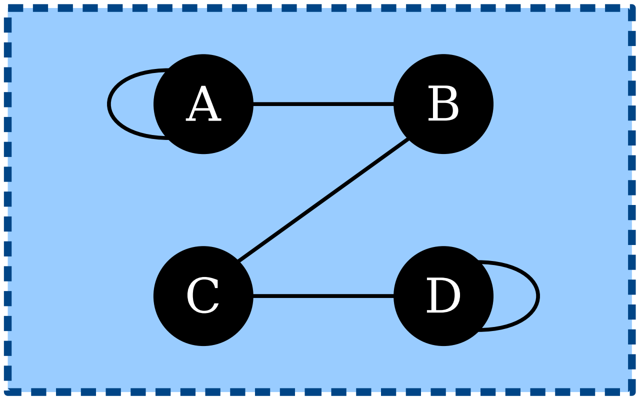

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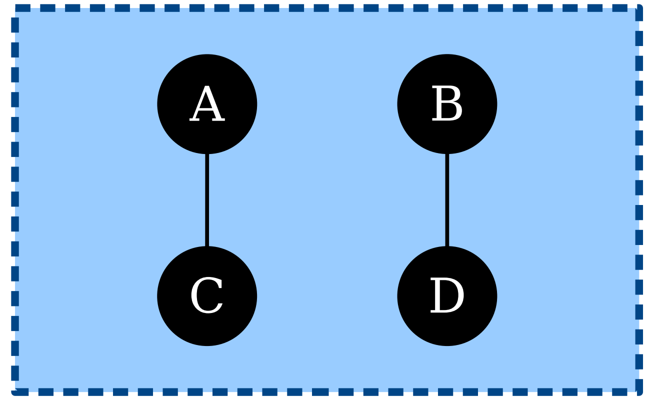

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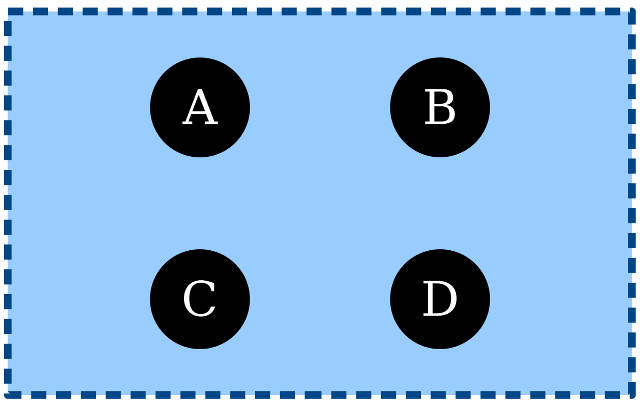

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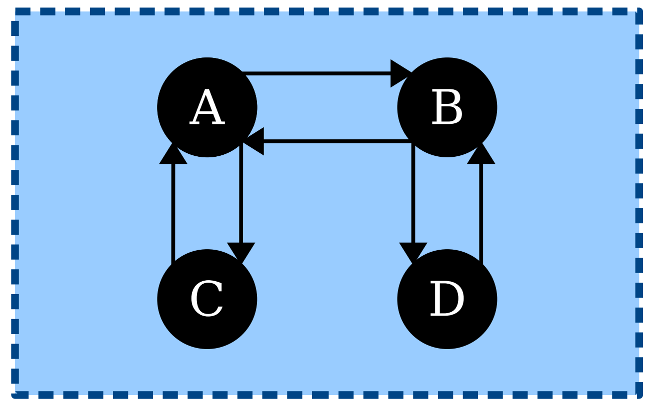
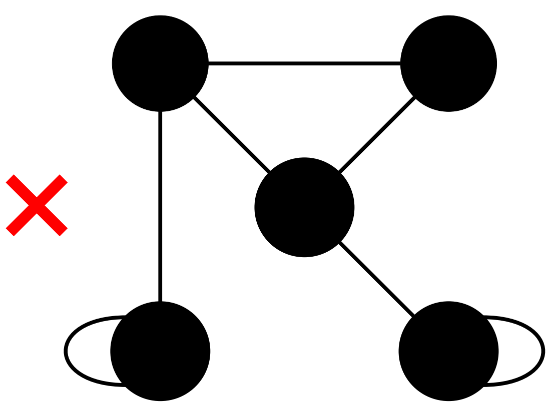


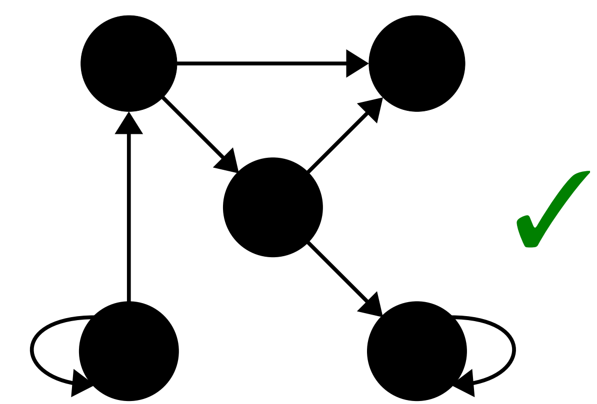
Which of these are drawings of undirected graphs?

Answer at
https://cs103.stanford.edu/pollev

Self-Loops

- An edge from a node to itself is called a ***self-loop***.
- In (undirected) graphs, self-loops are generally not allowed.
  - Can you see how this follows from the definition?
- In digraphs, self-loops are generally allowed unless specified otherwise.





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- An ***undirected graph*** is an ordered pair $G = (V, E)$, where
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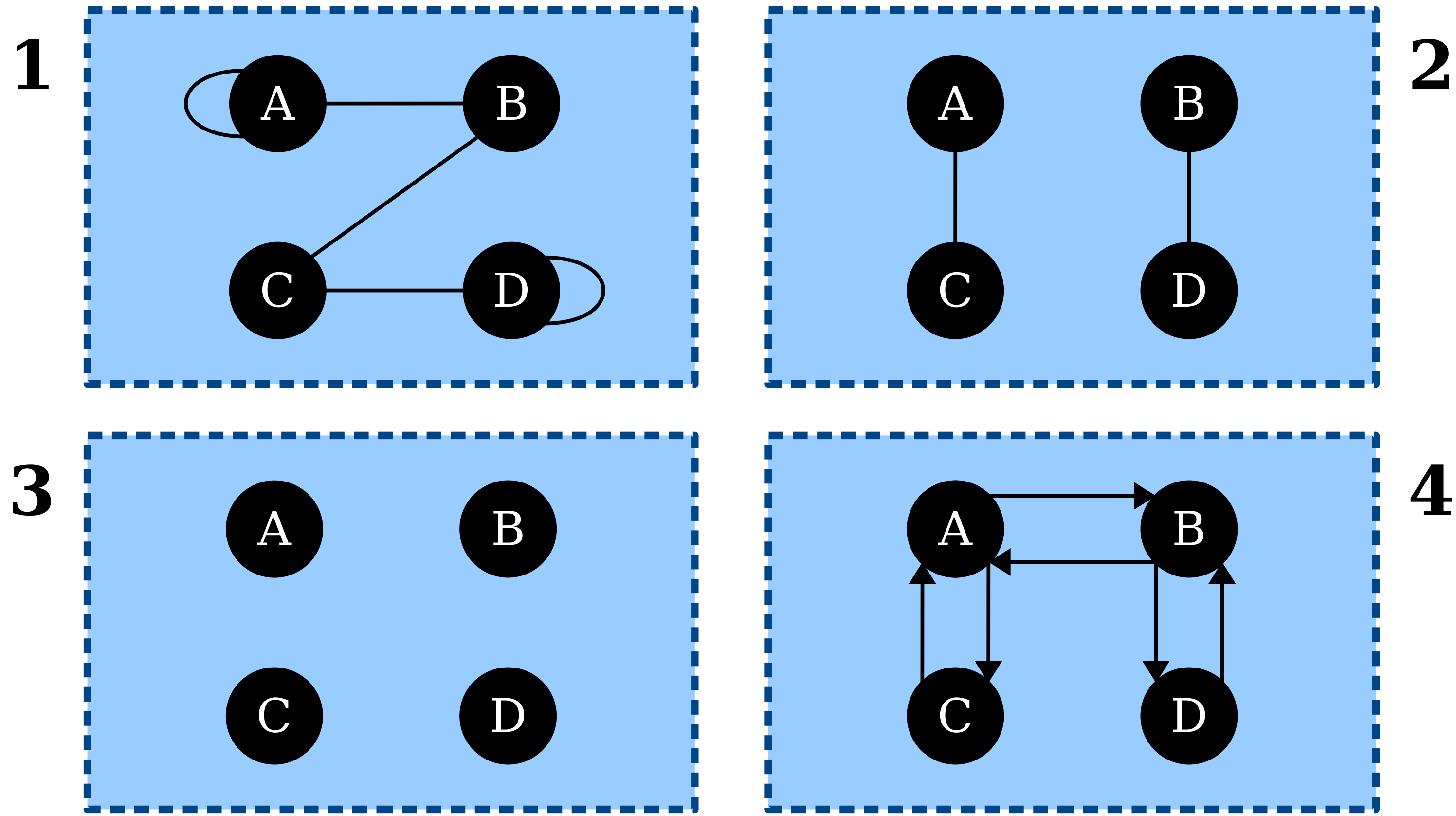

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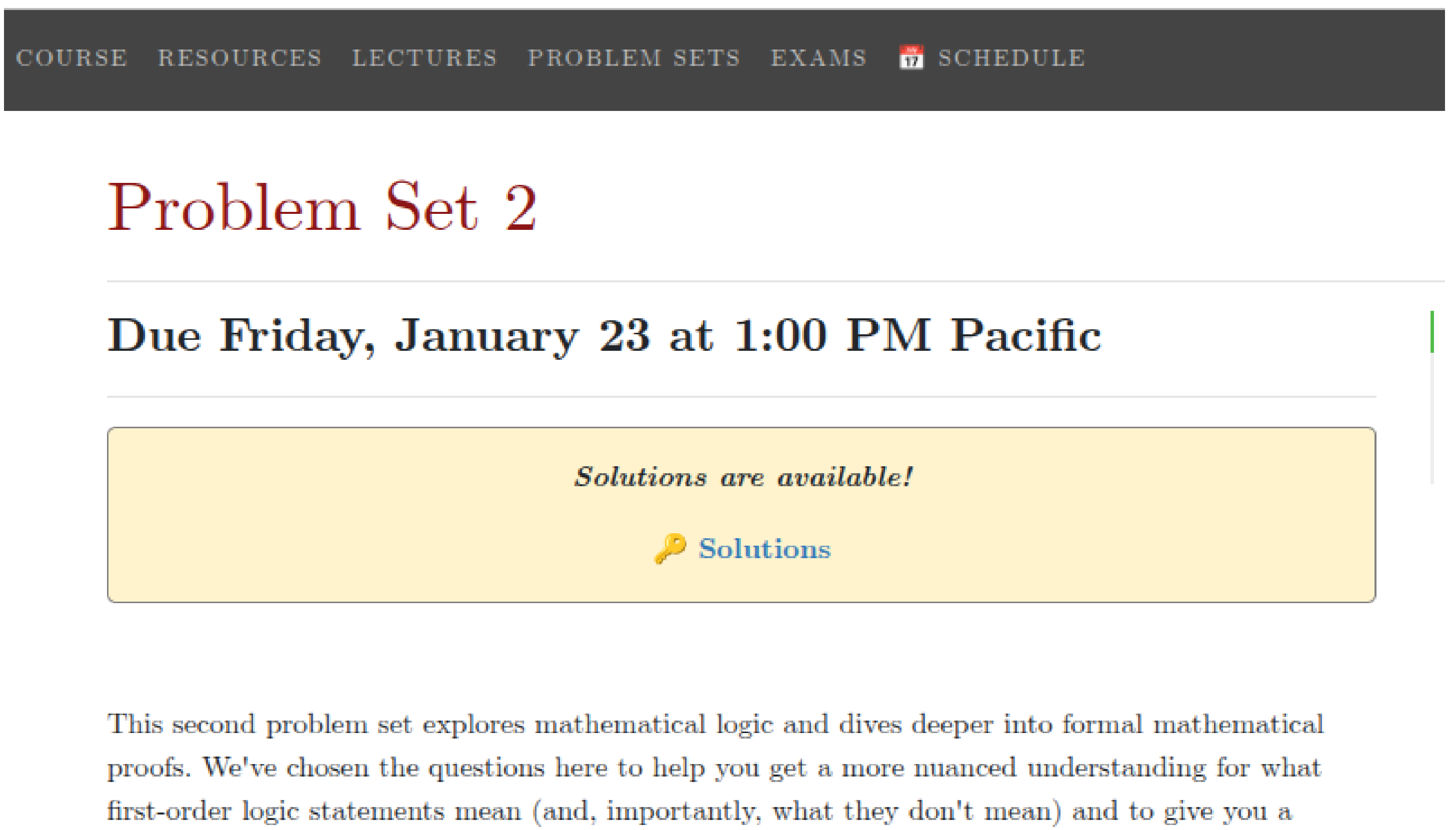
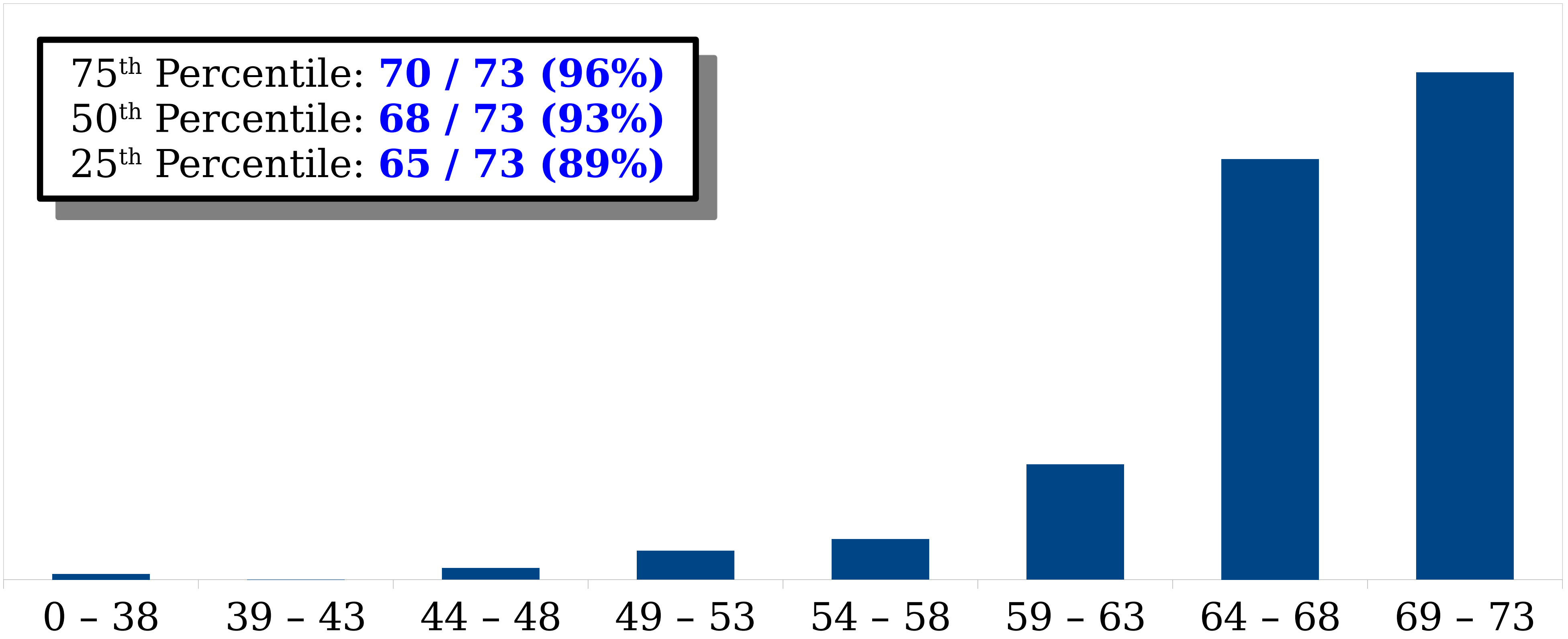
PS2 Solutions Released

- We don't release solutions to autograded problems.
- If you have any questions about those, ping us privately over EdStem or come talk to us at our office hours.



Problem Set Two Graded



PS3 Due Friday

- PS3 is due this Friday at 1:00PM.
    - Ask questions if you have them! That’s what we’re here for. You can ask on EdStem or in office hours.
    - If you haven’t yet started, please do so today. That will give you time to digest the concepts and read up on anything you’re still confused about.
    - Please tag problems. :)

Midterm Exam Logistics

- Our first midterm exam is next ***Tuesday, February 3rd***, from ***7–10 PM***. Locations vary.
- You're responsible for Lectures 00 – 05 and topics covered in PS1 – PS2, and the assume-prove table material from lectures 06 – 07. Later topics (functions forward) and problem sets (PS3 onward) won't be tested here, other than topics related to the assume-prove table. Exam problems may build on the written or coding components from the problem sets.
- The exam is closed-book, closed-computer, and limited-note. You can bring a double-sided, 8.5" × 11" sheet of notes with you to the exam, decorated however you'd like.

Midterm Exam Logistics

- Students with alternate exam arrangements: you should have heard from us with details. If you haven't, contact Anisha and Sean ASAP.
- We've posted an ***Exam Logistics*** page on the course website with full details and logistics.
- It will also include advice from former CS103 students about how to do well on the exams.
- Check it out – there will be tons of goodies on

Exam Day Logistics

- We'll have proctors in the room.
- We have assigned seating; see course website. These will be posted late this afternoon (Wednesday).
- Check out your seat in advance, and screenshot it!
- No phones, calculators, or other digital devices during the exam.
- No belongings near your seats (leave them in your dorms or at the front of the classroom).
- Must have Stanford student ID to turn in exam.

Review Session

- Two of our amazing CAs, Aditi and Anisha, will hold a review session this ***Thursday, January 29th, 5:30 – 7:30 PM***. More details on Ed!
  - The review session will be recorded.
- Come prepared to discuss any questions you may have.
- You'll get more out of this session if you have done some preliminary study first.

Participation Opt-Out

- Participation opt-out due Friday, 11:59 PM.
- This is a firm deadline.

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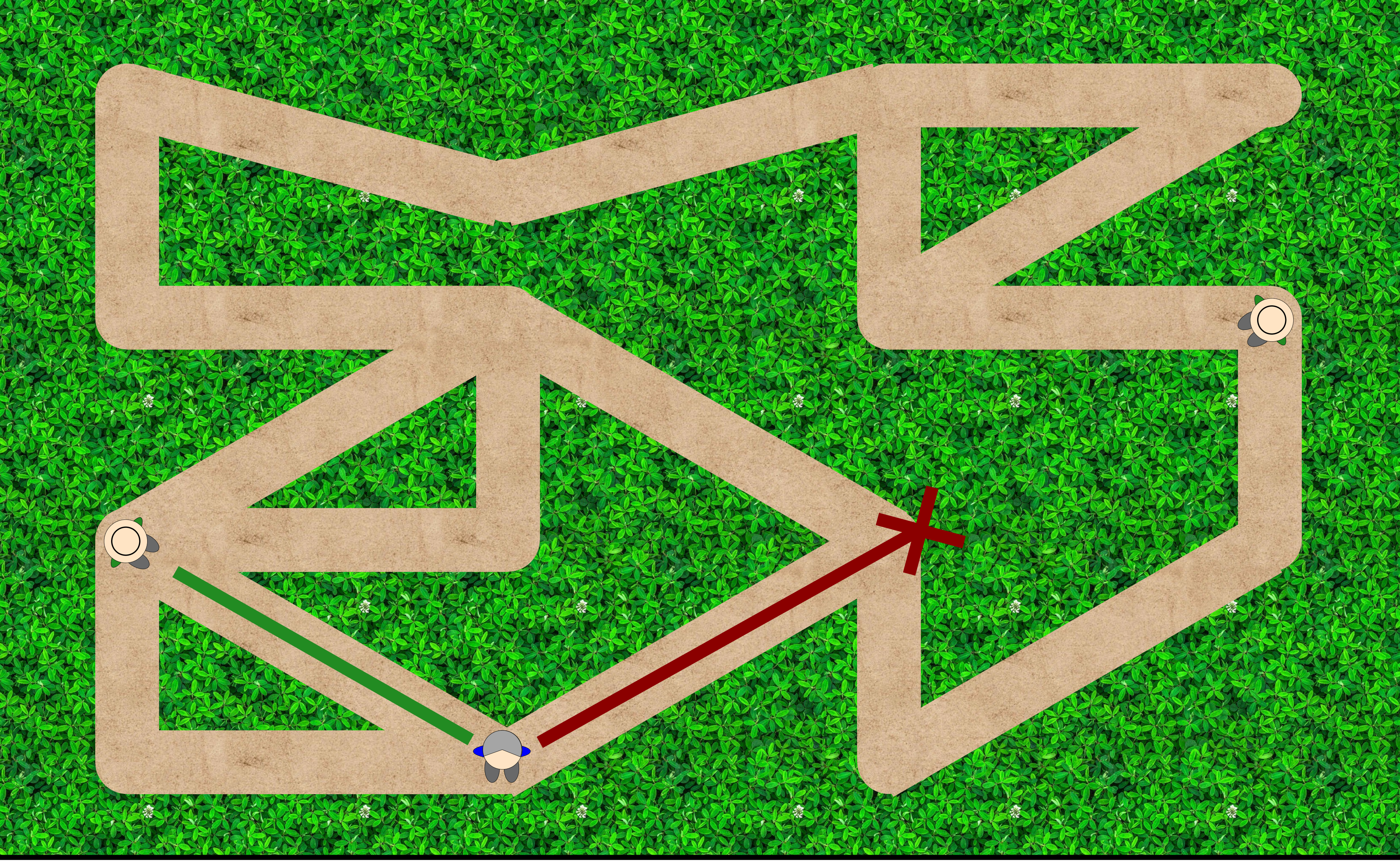
Vertex Covers

- Let $G = (V, E)$ be an undirected graph. A ***vertex cover*** of G is a set $C \subseteq V$ such that the following statement is true:

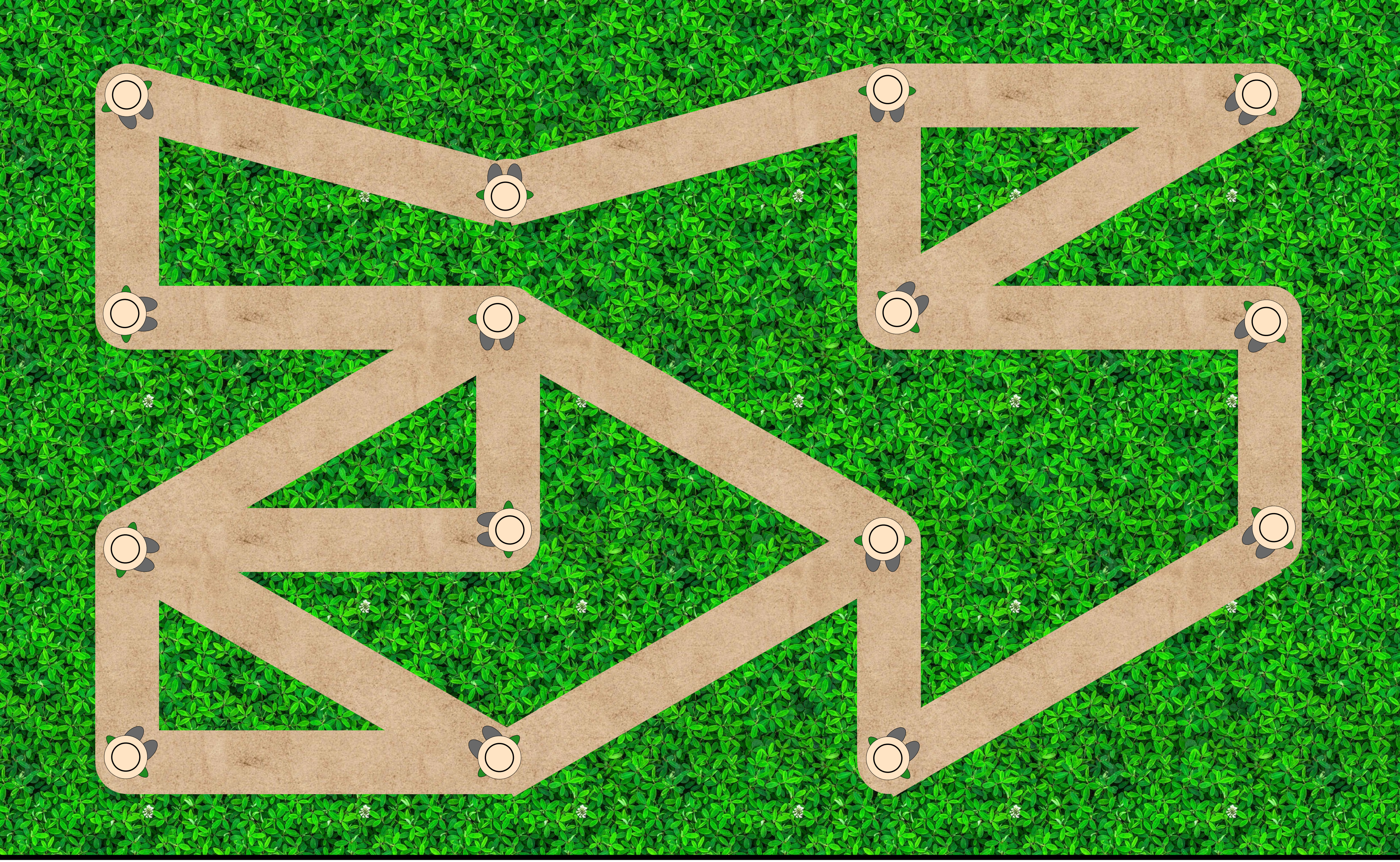
$$\forall u \in V.\ \forall v \in V.\ (\{u, v\} \in E \rightarrow (u \in C \lor v \in C))$$

("Every edge has at least one endpoint in C.")

- Intuitively speaking, a vertex cover is a set formed by picking at least one endpoint of each edge in the graph.
- Vertex covers have applications to placing streetlights/benches/security guards, as well as in gene sequencing, machine learning, and combinatorics.



Place park rangers in these forest trails so that
a hiker anywhere on a trail can see a park ranger.



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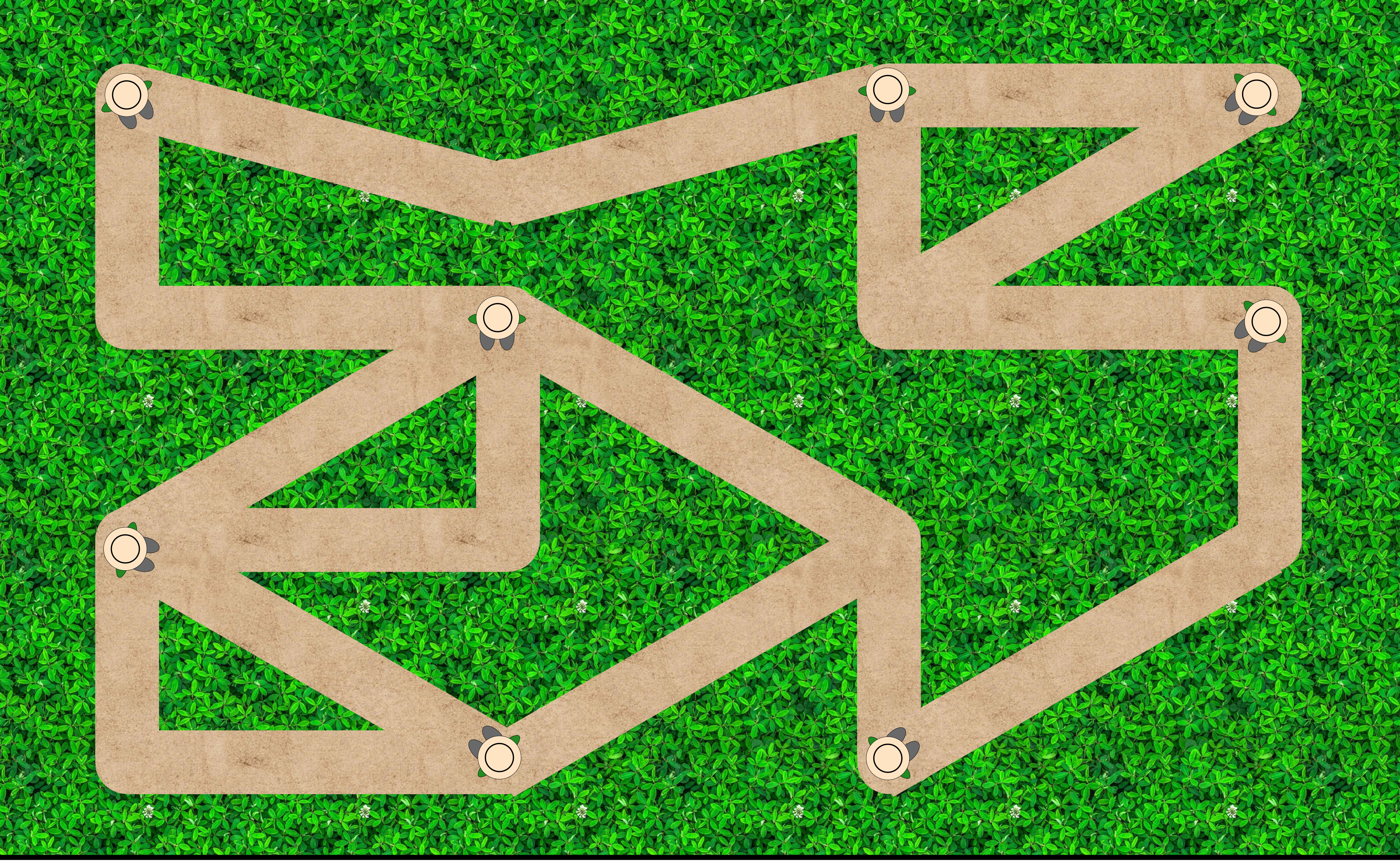

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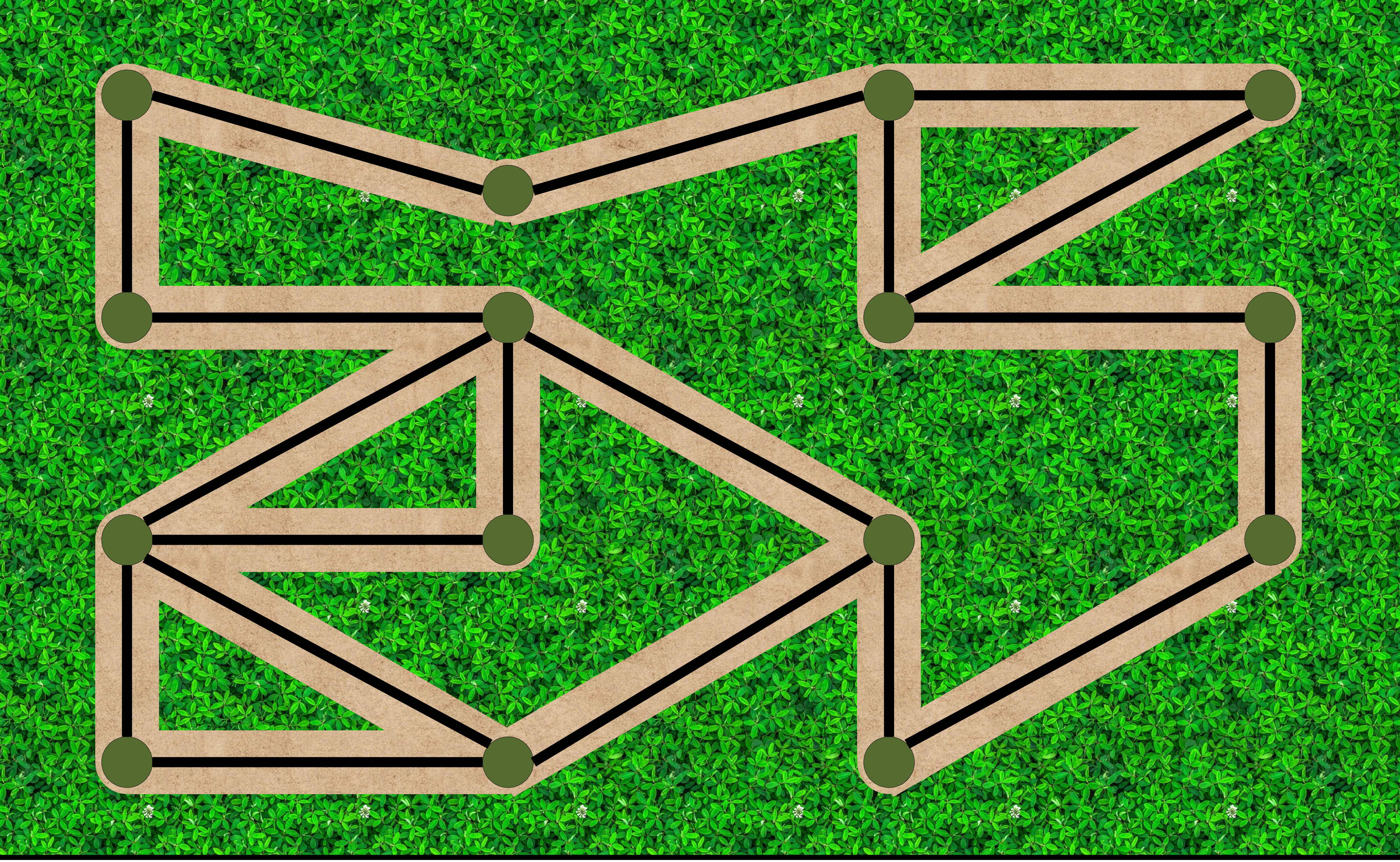

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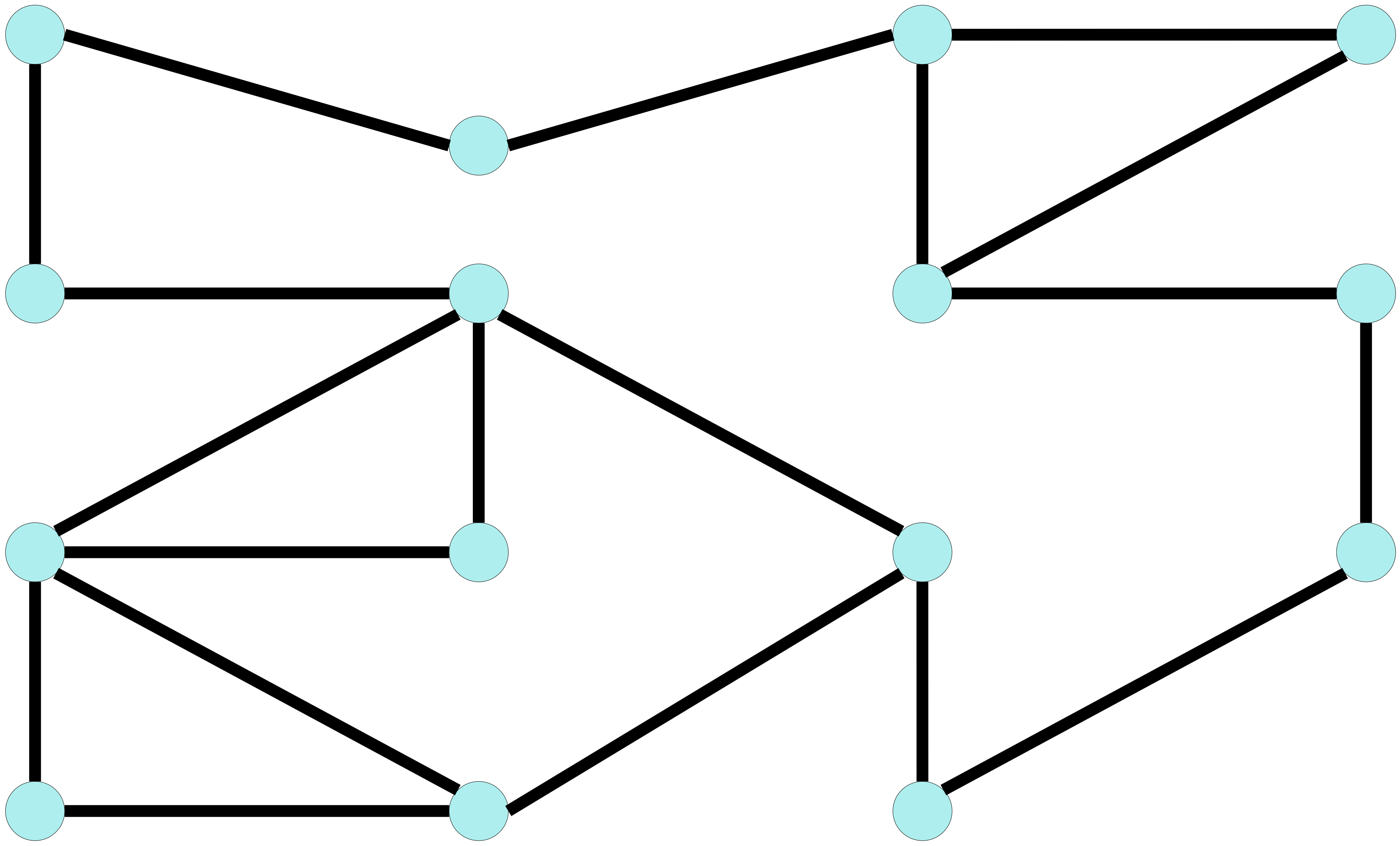

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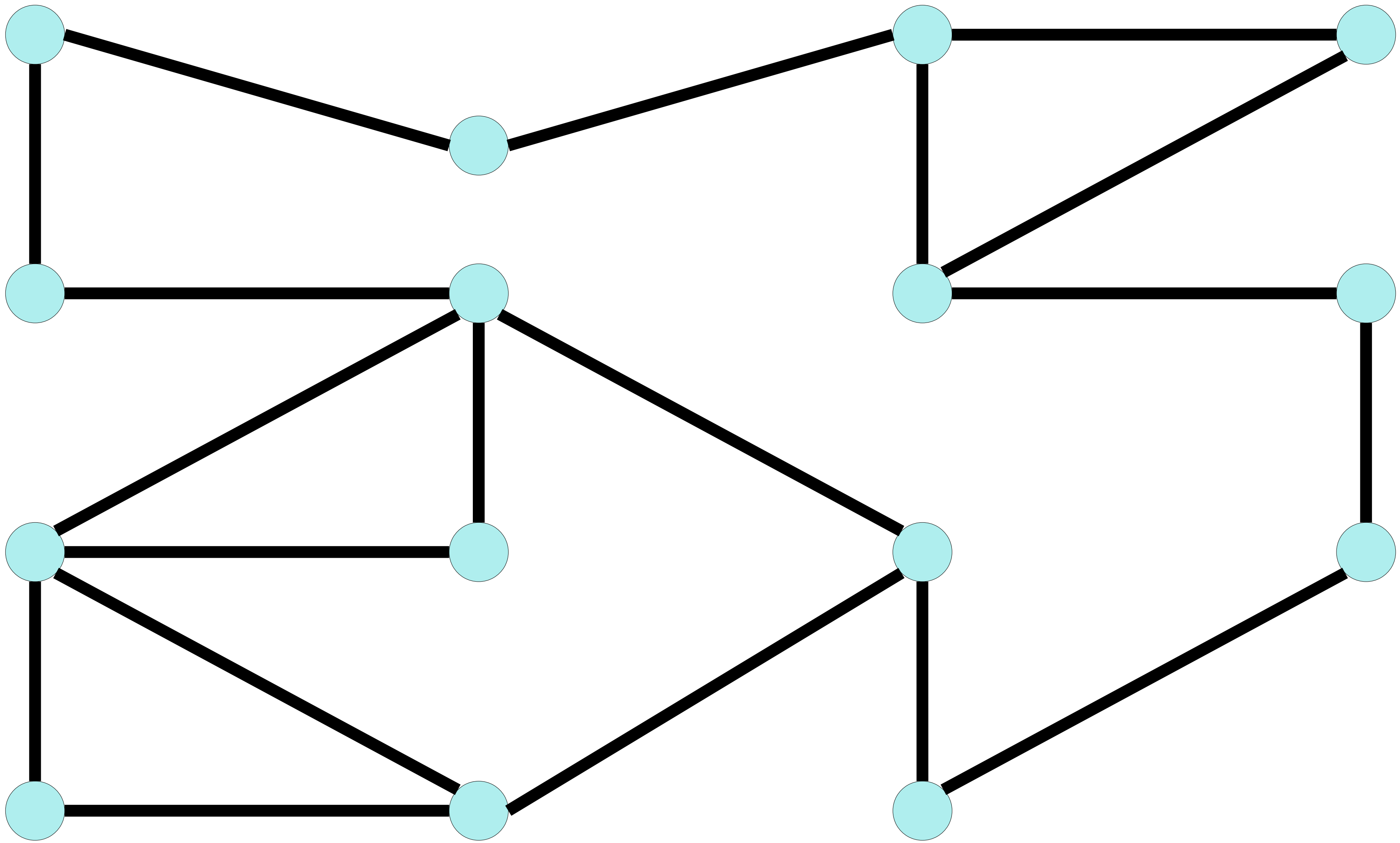
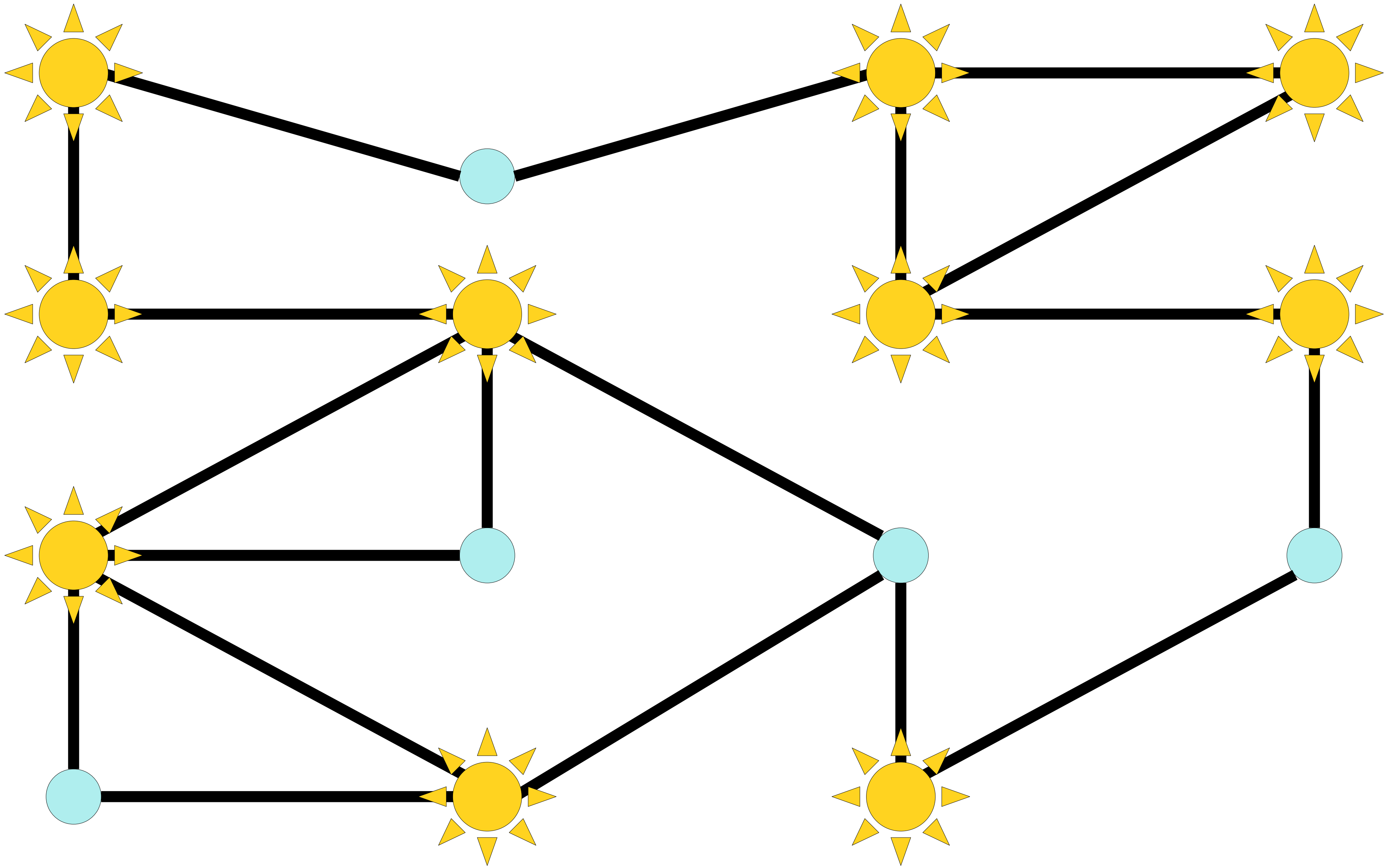


Choose at least one endpoint of each edge.



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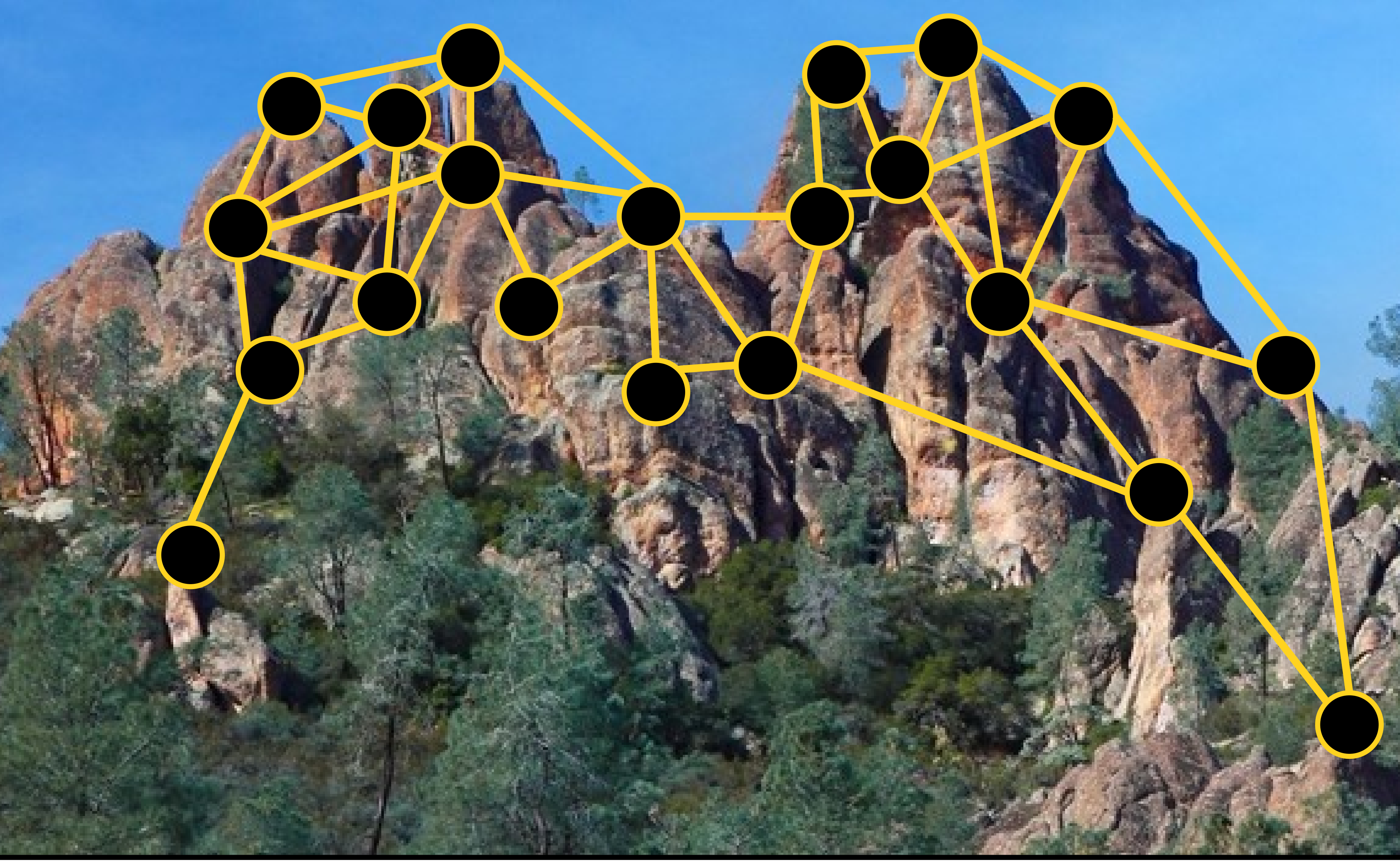
Independent Sets

- If $G = (V, E)$ is an (undirected) graph, then an ***independent set*** in G is a set $I \subseteq V$ such that

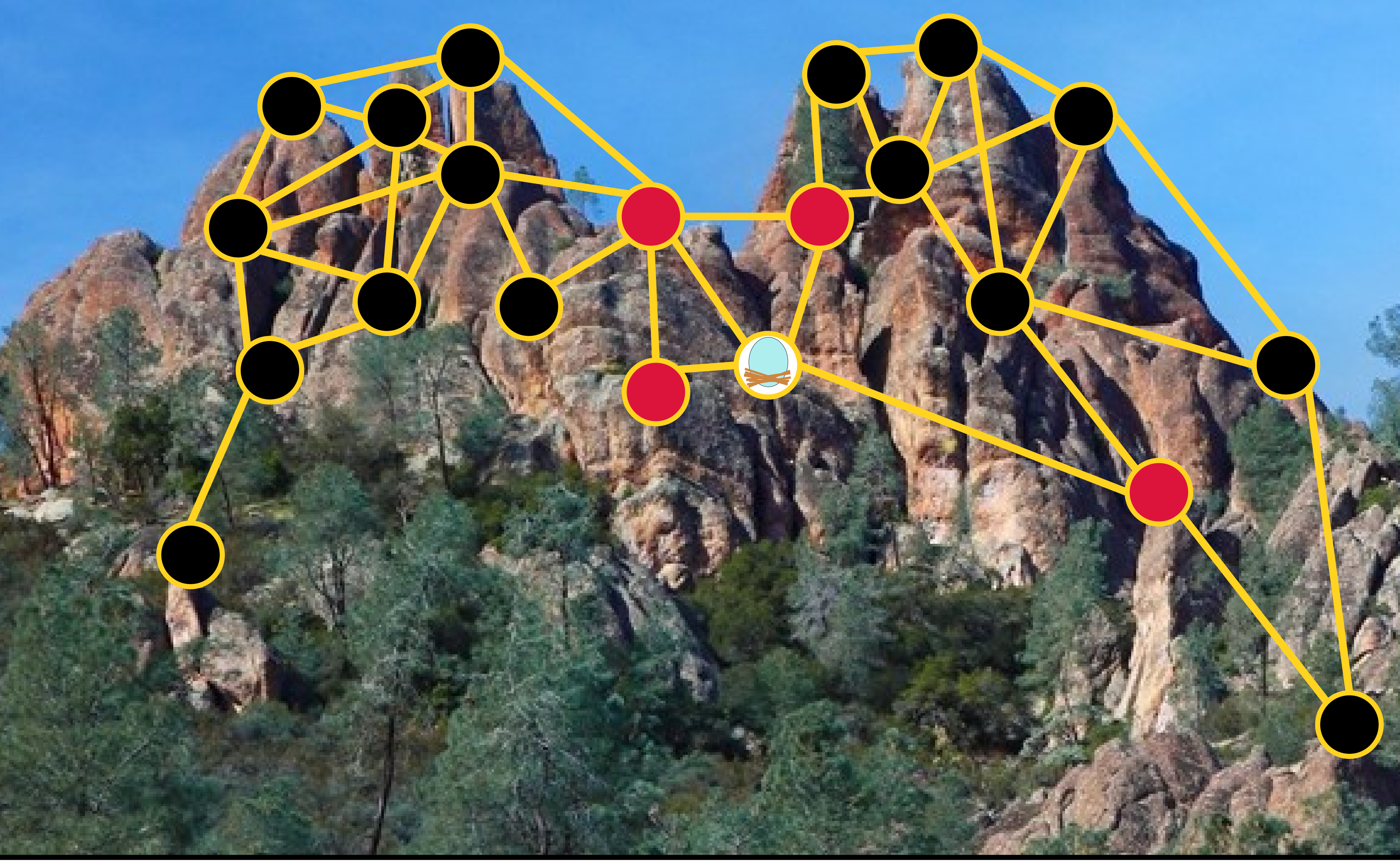
$$\forall x \in I.\ \forall y \in I.\ \{x, y\} \notin E.$$

("No two nodes in I are adjacent.")

- Independent sets have applications to resource optimization, conflict minimization, error-correcting codes, cryptography, and more.



Set up nests for the California condor. Condors are territorial and won’t nest if they can see other condors.



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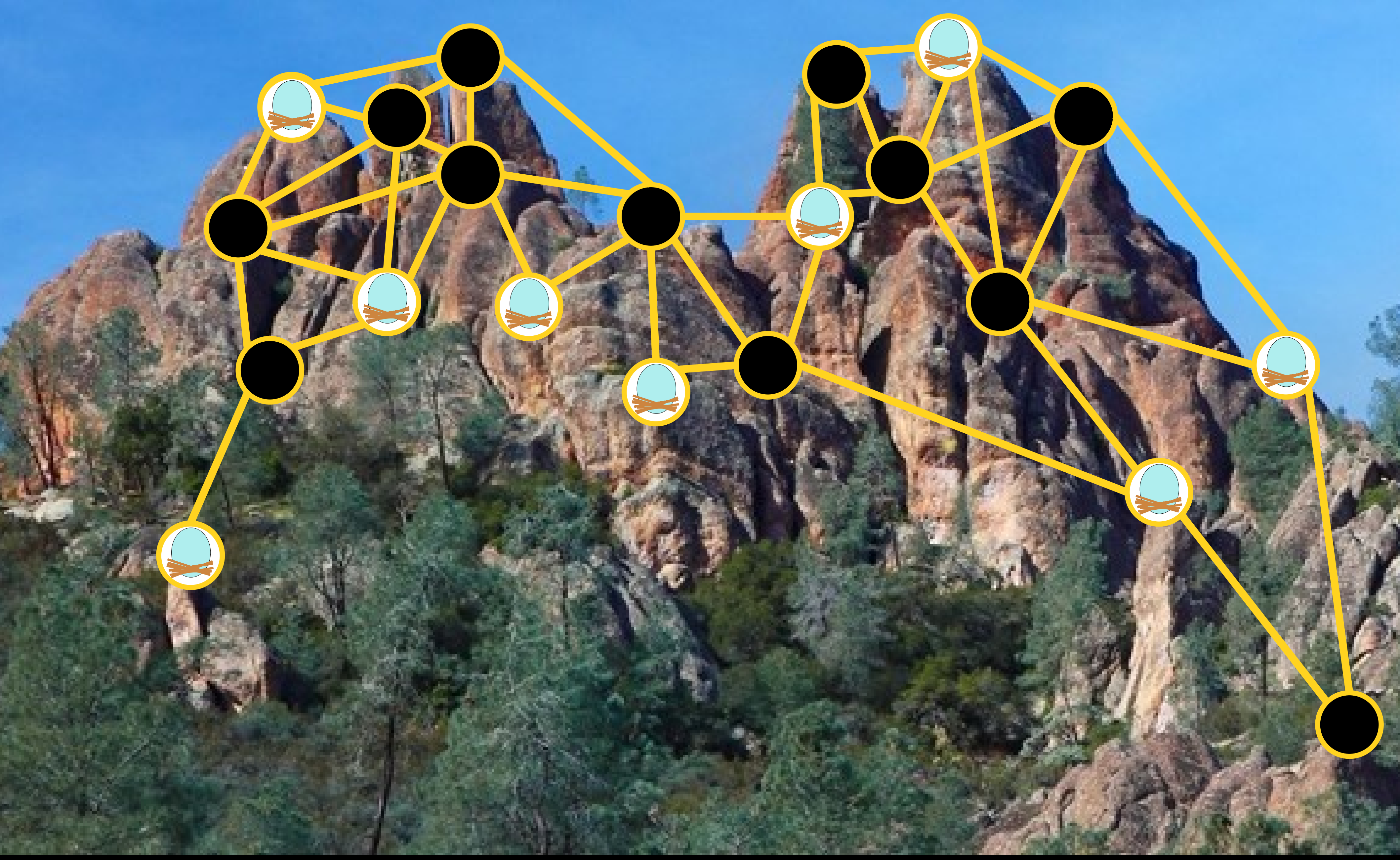

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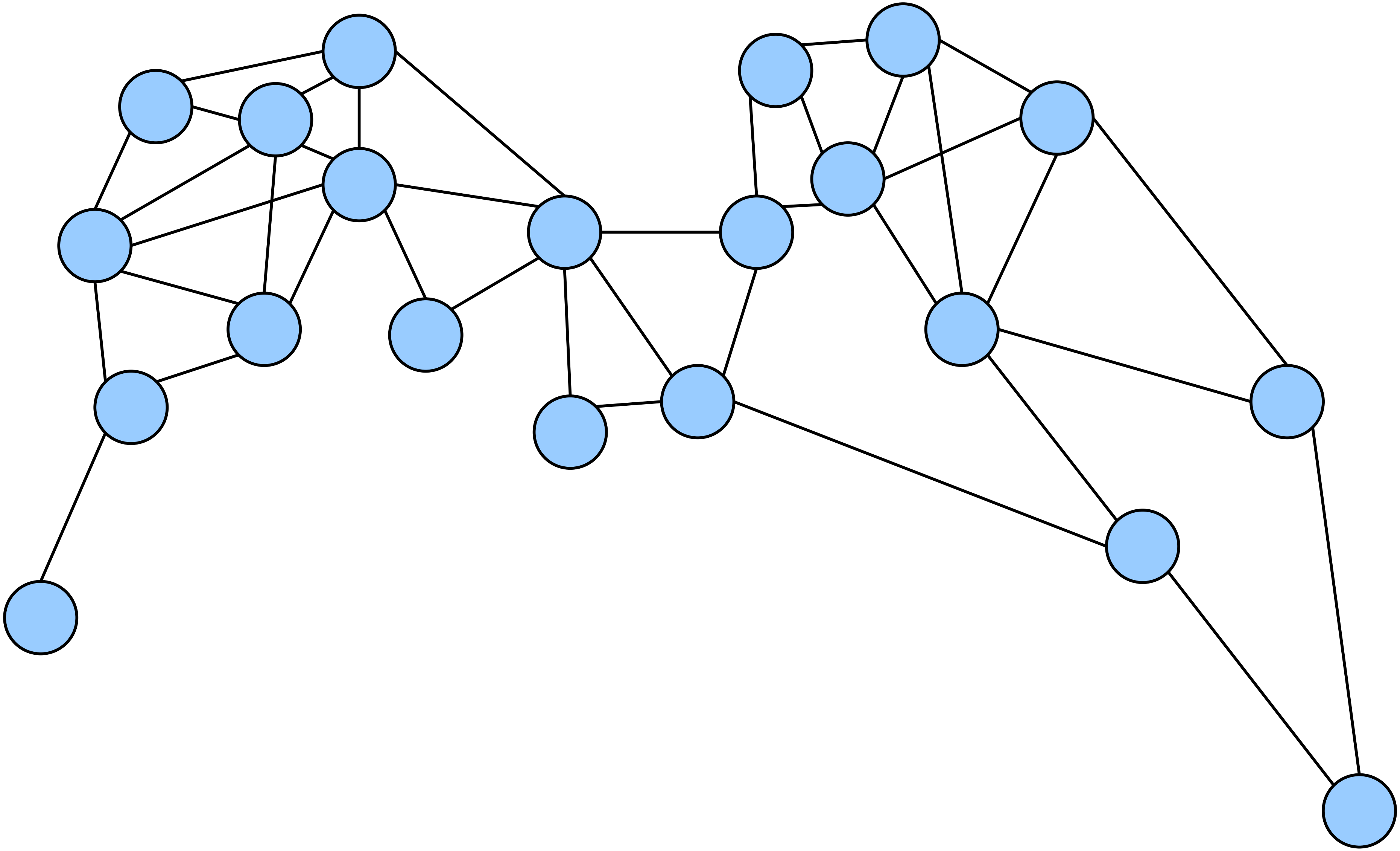

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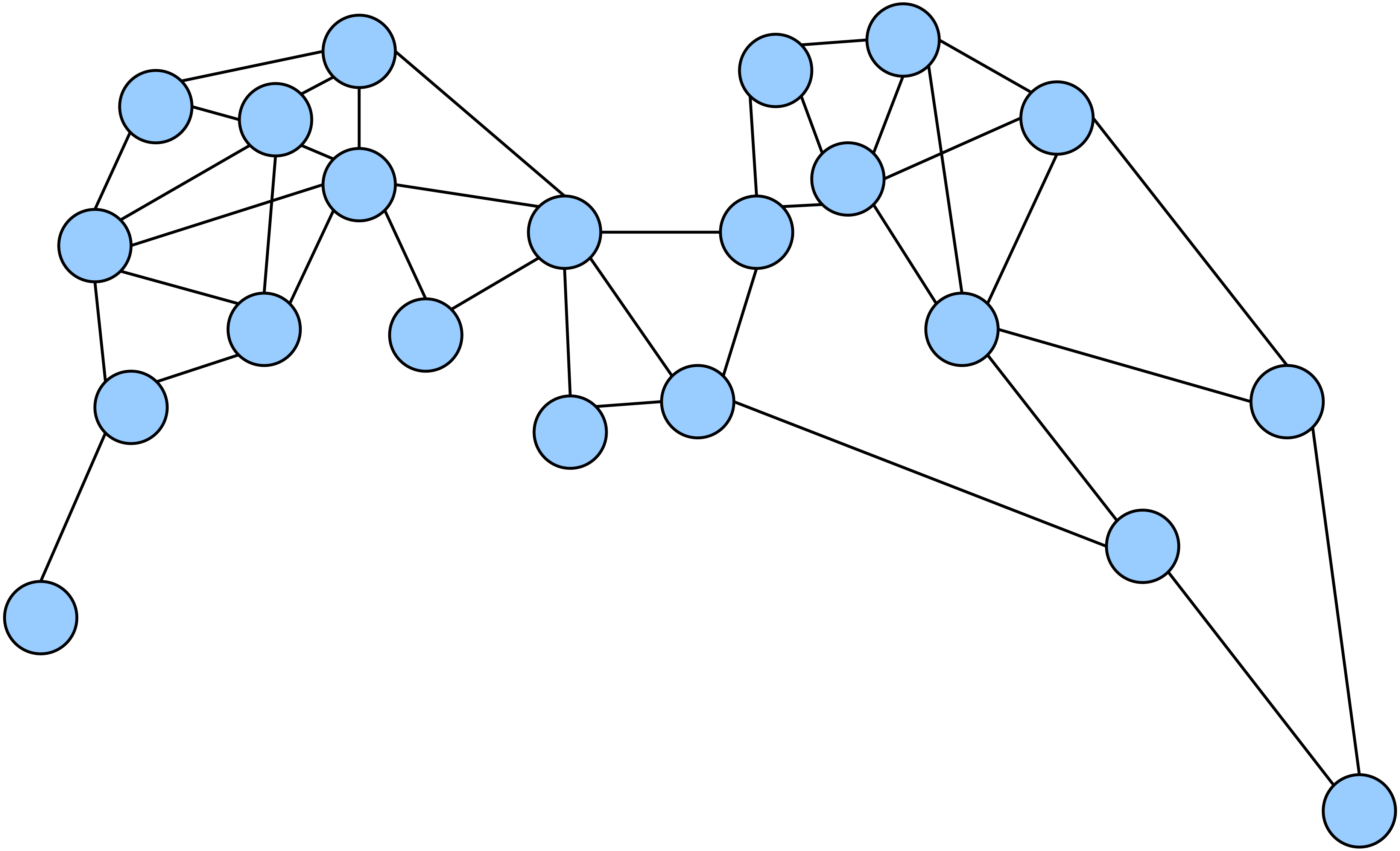

Choose a set of nodes, no two of which are adjacent.

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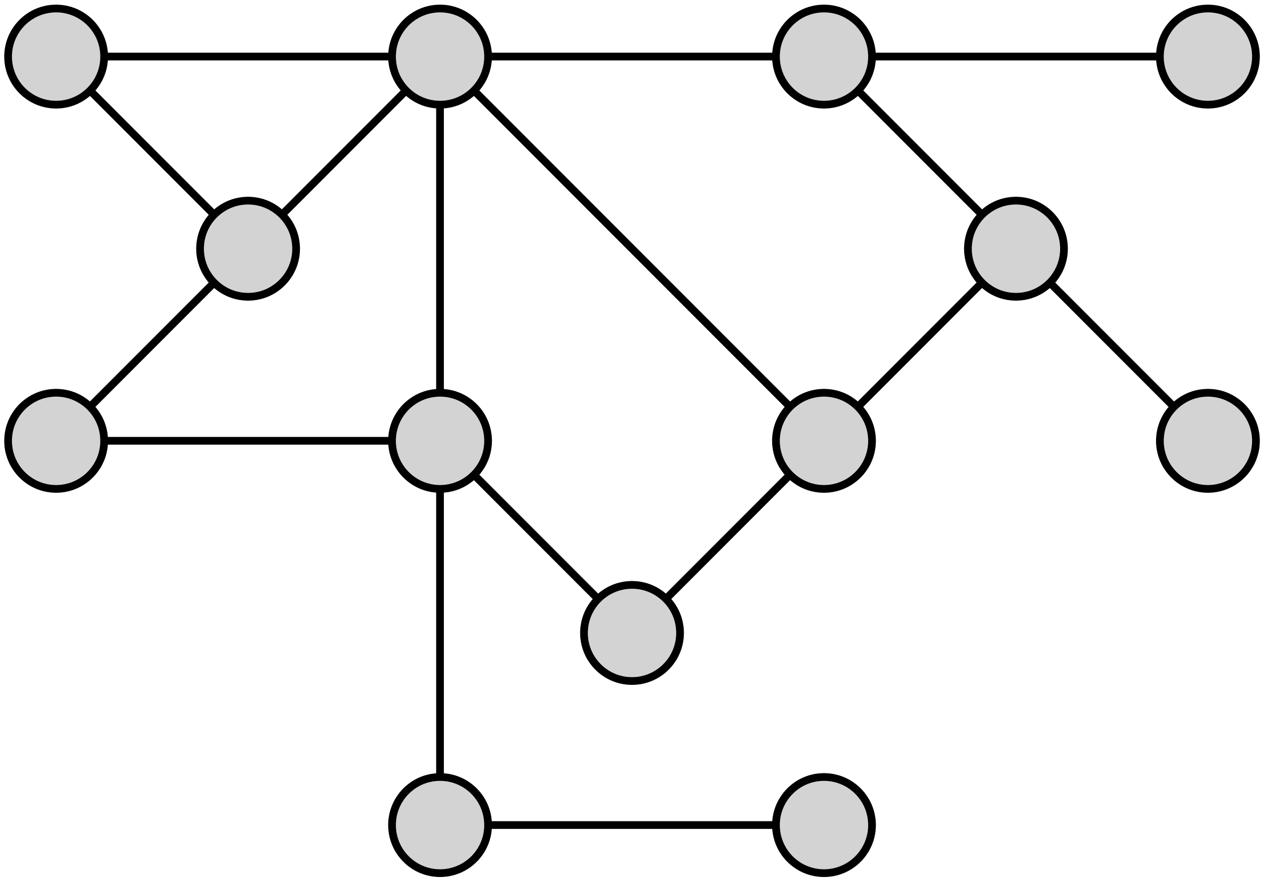
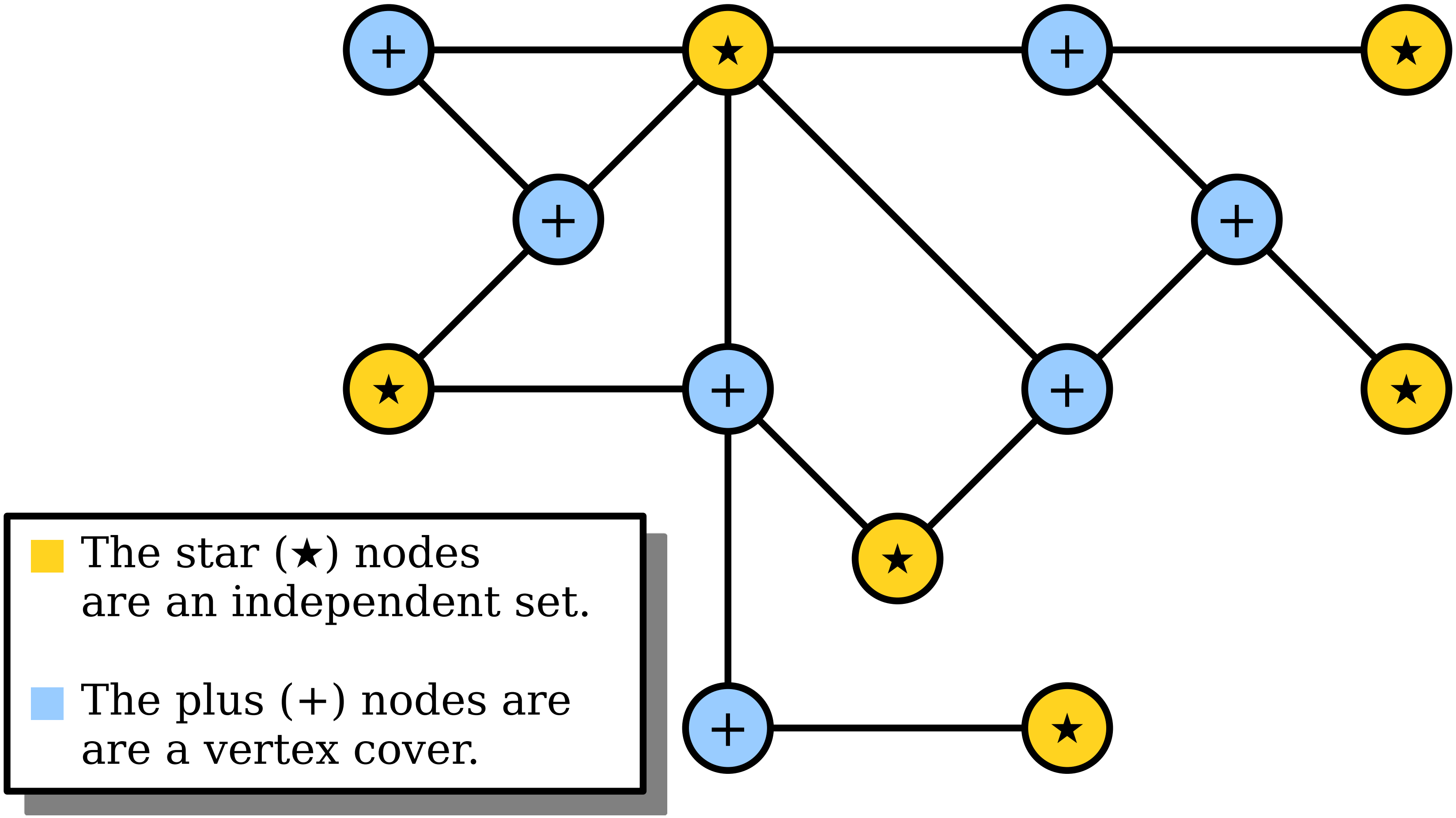
A Connection



Independent sets and vertex covers are related.



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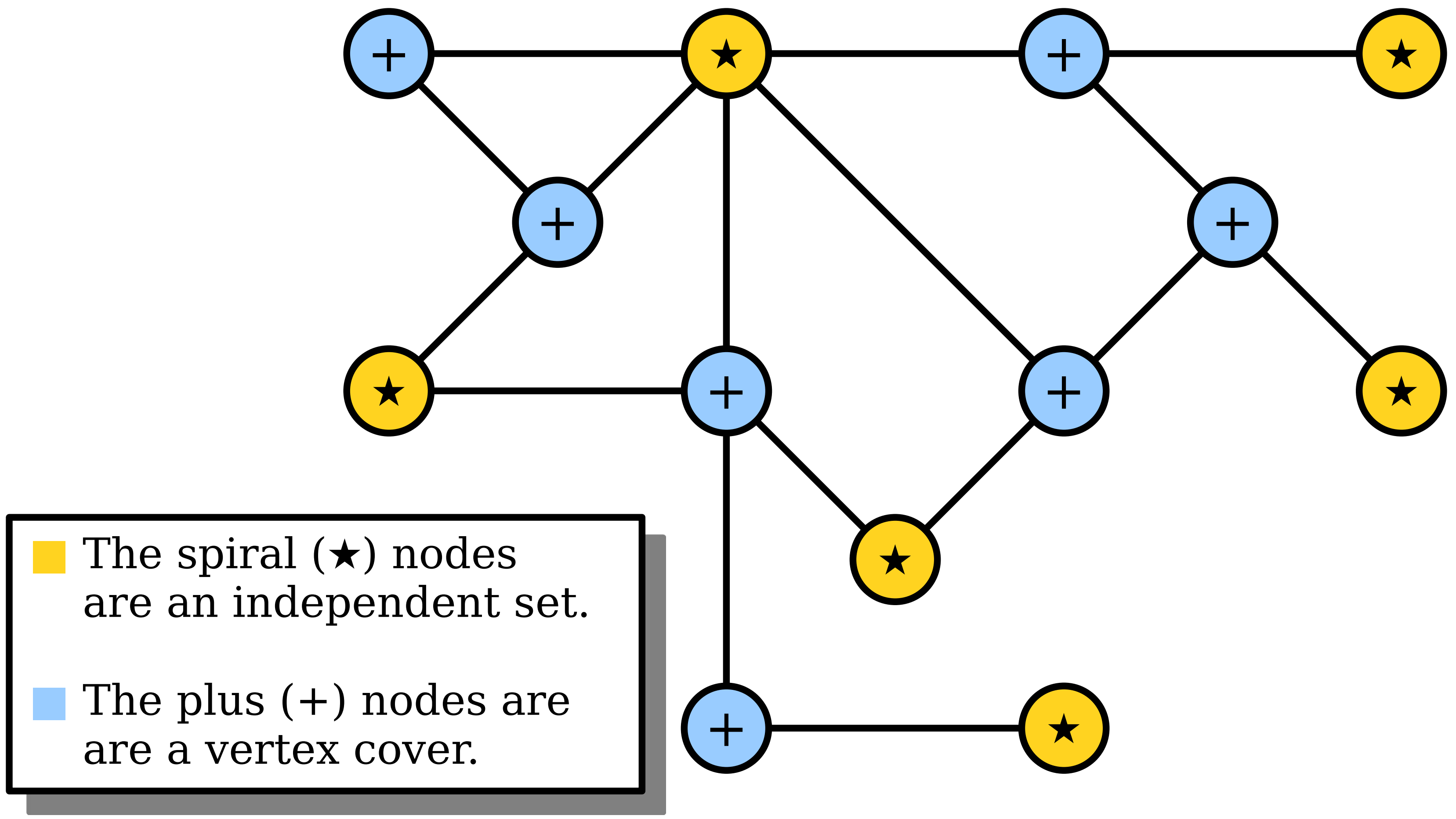

Theorem: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. Then C is a vertex cover of G if and only if $V - C$ is an independent set in G.

Lemma 1: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If C is a vertex cover of G, then $V - C$ is an independent set in G.

What We're Assuming	What We Need To Show
G is a graph.	$V - C$ is an independent set in G.
C is a vertex cover of G.	

$$\forall u \in V. \forall v \in V. (\{u, v\} \in E \to u \in C \lor v \in C)$$

$$\forall x \in V - C. \forall y \in V - C. \{x, y\} \notin E.$$

We're assuming a universally-quantified statement. That means we *don't do anything right now* and instead wait for an edge to present itself.

We need to prove a universally-quantified statement. We'll ask the reader to pick arbitrary choices of x and y for us to work with.

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What We're Assuming	What We Need To Show
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C is a vertex cover of G.	$\forall x \in V - C.$
$\forall u \in V. \forall v \in V. (\{u, v\} \in E \to$ $u \in C \lor v \in C$ $)$	$\forall y \in V - C.$
$x \in V$ and $x \notin C$.	$\{x, y\} \notin E.$
$y \in V$ and $y \notin C$.	

Lemma 1: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If C is a vertex cover of G, then $V - C$ is an independent set in G.

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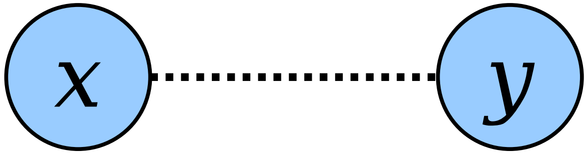

If this edge exists, at least one of x and y is in C.

Lemma 1: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If C is a vertex cover of G, then $V - C$ is an independent set of G.

Proof: Assume C is a vertex cover of G. We need to show that $V - C$ is an independent set of G. To do so, pick any nodes $x, y \in V - C$; we will show that $\{x, y\} \notin E$.

Suppose for the sake of contradiction that $\{x, y\} \in E$. Because $x, y \in V - C$, we know that $x \notin C$ and $y \notin C$. However, since C is a vertex cover of G and $\{x, y\} \in E$, we also see that $x \in C$ or $y \in C$, contradicting our previous statement.

We've reached a contradiction, so our assumption was wrong. Therefore, we have $\{x, y\} \notin E$, as required. ■

Lemma 2: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If C is not a vertex cover of G, then $V - C$ is not an independent set of G.

See appendix for this proof!

Be sure to check this one out and follow through with the negations of the statements above.

Finding an IS or VC

- The previous theorem means that finding a large IS in a graph is equivalent to finding a small VC.
  - If you've found one, you've found the other!
- ***Open Problem:*** Design an algorithm that, given an n-node graph, finds either the largest IS or smallest VC "efficiently," where "efficiently" means "in time $O(n^k)$ for some $k \in \mathbb{N}$."
  - There's a $1,000,000 bounty on this problem – we'll see why in Week 10.

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Recap for Today

- A ***graph*** is a structure for representing items that may be linked together. ***Digraphs*** represent that same idea, but with a directionality on the links.
- Graphs can't have ***self-loops***; digraphs can.
- ***Vertex covers*** and ***independent sets*** are useful tools for modeling problems with graphs.
- The complement of a vertex cover is an independent set, and vice-versa.

Next Time

- ***Paths and Trails***
  - Walking from one point to another.
- ***Local Area Networks***
  - The building blocks of the internet.
- ***Trees***
  - A fundamental class of graphs.

Appendix

Lemma 2: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If C is not a vertex cover of G, then $V - C$ is not an independent set of G.

Taking Negations

- What is the negation of this statement, which says "C is a vertex cover?"

$$
\begin{aligned}
&\forall u \in V.\ \forall v \in V.\ (\{u, v\} \in E \to \\
&\qquad u \in C \quad \lor \quad v \in C \\
&)
\end{aligned}
$$

Taking Negations

- What is the negation of this statement, which says "*C* is a vertex cover?"

$$\exists u \in V.\ \exists v \in V.\ (\{u, v\} \in E \land$$
$$u \notin C \quad \land \quad v \notin C$$
$$)$$

- This says "there is an edge where both endpoints aren't in *C*."

Taking Negations

- What is the negation of this statement, which says "$V - C$ is an independent set?"

$$\forall u \in V - C.\ \forall v \in V - C.\ \{u, v\} \notin E$$

Taking Negations

- What is the negation of this statement, which says "$V - C$ is an independent set?"

$$\exists u \in V - C.\ \exists v \in V - C.\ \{u, v\} \in E$$

- This says "there are two adjacent nodes in $V - C$."

Lemma 2: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If C is not a vertex cover of G, then $V - C$ is not an independent set in G.

What We're Assuming	***What We Need To Show***
G is a graph.	$V - C$ is not an ind. set in G.
C is a not a vertex cover of G.	$\exists x \in V - C.$ $\exists y \in V - C.$ $\{x, y\} \in E.$
$\exists u \in V.\ \exists v \in V.\ (\{u, v\} \in E\ \land$ $u \notin C \quad \land \quad v \notin C$ $)$	
We're assuming an existentially-quantified statement, so we'll *immediately* introduce variables u and v.	We're proving an existentially-quantified statement, so we *don't* introduce variables x and y. We're on a scavenger hunt!

Lemma 2: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If C is not a vertex cover of G, then $V - C$ is not an independent set in G.

What We're Assuming	***What We Need To Show***
G is a graph.	$V - C$ is not an ind. set in G.
C is a not a vertex cover of G.	$\exists x \in V - C$.
$u \in V - C$.	$\exists y \in V - C$.
$v \in V - C$.	$\{x, y\} \in E$.
$\{u, v\} \in E$.	

Any ideas about what we should pick x and y to be?

Lemma 2: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If C is not a vertex cover of G, then $V - C$ is not an independent set of G.

Proof: Assume C is not a vertex cover of G. We need to show that $V - C$ is not an independent set of G.

Since C is not a vertex cover of G, we know that there exists nodes $x, y \in V$ where $\{x, y\} \in E$, where $x \notin C$, and where $y \notin C$. Because $x \in V$ and $x \notin C$, we know that $x \in V - C$. Similarly, we see that $y \in V - C$.

This means that $\{x, y\} \in E$, that $x \in V - C$, and that $y \in V - C$, and therefore that $V - C$ is not an independent set of G, as required. ■